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## Homework 4 Solutions /

4.6

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{(2n)^2} - \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \\ \left[ \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \right] &= \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

$$\begin{aligned} 4.8 \text{(a)} \sum_{m=0}^{N-1} e^{2\pi i m/N} &= 1 + e^{2\pi i/N} + e^{4\pi i/N} + \dots + e^{2\pi i(N-1)/N} \\ &\stackrel{(i)}{=} \sum_{n=0}^{N-1} x^n \quad x = e^{2\pi i/N} \\ &= \frac{1-x^{N+1}}{1-x} = \frac{1-e^{2\pi i/N}}{1-e^{2\pi i/N}} = 1 \end{aligned}$$

Note:  $\sum_{m=0}^{N-1} e^{2\pi i m/N} = 0$

(ii)  $\sum_{m=0}^{N-1} e^{4\pi i m/N} = \frac{1-e^{4\pi i(N+1)/N}}{1-e^{4\pi i/N}} = 1$

(iii)  $\sum_{m=0}^{N-1} e^{2\pi i m/N} x^m = \frac{1-e^{2\pi i(N+1)/N} x^{N+1}}{1-e^{2\pi i/N} x} = \frac{1-e^{2\pi i/N} x^{N+1}}{1-e^{2\pi i/N} x}$

(b)  $\sum_{m=0}^{N-1} \cos \frac{2\pi m}{N} - \sum_{m=0}^{N-1} \cos \frac{4\pi m}{N} = \operatorname{Re} \sum_{m=0}^{N-1} e^{2\pi i m/N} - \operatorname{Re} \sum_{m=0}^{N-1} e^{4\pi i m/N}$   
 $= 1 - 1 = 0 \checkmark$

$\sum_{m=0}^{N-1} z^m \sin \frac{2\pi m}{N} = \operatorname{Im} \sum_{m=0}^{N-1} e^{2\pi i m/N} z^m = \operatorname{Im} \left( \frac{1-e^{2\pi i/N} z^{N+1}}{1-e^{2\pi i/N} z} \right)$

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N=3

$$\begin{aligned} \sum_{m=0}^N 2^m \sin \frac{2\pi m}{N} &= \operatorname{Im} \left[ \frac{1 - e^{2\pi i/3}}{1 - 2e^{2\pi i/3}} 2^4 \right] \\ &= \operatorname{Im} \left[ \frac{1 - 16(\cos 2\pi/3 + i \sin 2\pi/3)}{1 - 2(\cos 2\pi/3 + i \sin 2\pi/3)} \right] \\ &= \operatorname{Im} \left[ \frac{9 - i8\sqrt{3}}{2 - i\sqrt{3}} \right] = \operatorname{Im} \left[ \frac{(9 - i8\sqrt{3})(2 + i\sqrt{3})}{2^2 + 3} \right] = \\ &= \frac{-16\sqrt{3} + 9\sqrt{3}}{7} = -\frac{\sqrt{3}}{7} \end{aligned}$$

4,10] a)  $\sum_{n=1}^{\infty} \frac{2 \sin n\pi}{n(n+1)} \leq \sum_{n=1}^{\infty} \frac{2}{n(n+1)} < \sum_{n=1}^{\infty} \frac{2}{n^2}$  [converges]

b)  $\sum_{n=1}^{\infty} \frac{2}{n^2} < 1 + \int_1^{\infty} dx \frac{2}{x^2}$  [converges]

c)  $\sum_{n=1}^{\infty} \frac{1}{2n^{1/2}}$  [diverges]:  $\int_1^{\infty} \frac{dx}{x^{1/2}} = \infty$

d)  $\sum_n (-1)^n \frac{(n^2)^{1/2}}{n!} = \sum_n (-1)^n b_n$

$b_{n+1} < b_n$  for all  $n$

$b_n \rightarrow 0$  as  $n \rightarrow \infty$  so

alternating sign series [converges]

e)  $\sum_{n=k}^{\infty} \frac{n^p}{n!}$  ratio test  $\frac{(n+1)^p}{n^p} \frac{n!}{(n+1)!} \cdot \left(1 + \frac{1}{n}\right)^p \xrightarrow[n \rightarrow \infty]{} 0$

$\Rightarrow$  [Converges]

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$$4.14 \quad \text{ratio test} \quad \sum_{n=1}^{\infty} x^{n/2} e^{-n}$$

$$\frac{b_{n+1}}{b_n} = \frac{x^{\frac{n+1}{2}} e^{-(n+1)}}{x^{\frac{n}{2}} e^{-n}}$$

$\xrightarrow{n \rightarrow \infty} x^{1/2} e^{-1}$

$$x < e^2$$

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$$4.16 \boxed{\sum_{n=1}^{\infty} f(n) \leq \int_1^{\infty} dx f(x) + f(1)}$$

$$f(x) = \frac{1}{x^p} \quad \sum_{n=1}^{\infty} n^{-p} = \zeta(p) \leq \int_1^{\infty} dx x^{-p} + 1$$

$$\zeta(p) \leq \left| \frac{1}{1-p} x^{1-p} \right|_1^{\infty} + 1$$

$$\leq \frac{1}{p-1} + 1 = \frac{p}{p-1}$$

$$\boxed{\zeta(p) \leq \frac{p}{p-1}}$$

$$4.18 \boxed{S = \sum_{n=1}^{\infty} \sum_{r=1}^n \frac{1}{r^2(n+1-r)^2} = \sum_{n=1}^{\infty} \sum_{r=1}^n O(n-r+\epsilon) \frac{1}{r^2(n+1-r)^3}}$$

$$= \sum_{r=1}^{\infty} \sum_{n=r}^{\infty} \frac{1}{r^2(n+1-r)^3}$$

$$n = j+r-1$$

$$= \sum_{r=1}^{\infty} \sum_{j=0}^{\infty} \frac{1}{r^2} \frac{1}{j^3} = \zeta(2) \zeta(3)$$

From the bound in 4.16

$$S < \frac{2}{2-1} \cdot \frac{3}{3-1} = 3$$

$$\zeta(2) = \frac{\pi^2}{6} \quad \zeta(3) = \text{Apery's constant}$$

$$= 1.20206$$

$$\zeta(2) \zeta(3) = 1.9773$$

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$$4.20 \quad \sum_{n=1}^{\infty} \frac{(-)^{n+1} x^{2n}}{(2n-1)!} = x^2 - \frac{x^4}{3!} + \frac{x^6}{9!} - \frac{x^8}{7!} + \dots$$

$$= x \sin x$$

$$\text{a)} \frac{d}{dx} \sum_{n=1}^{\infty} \frac{(-)^{n+1} x^{2n}}{(2n-1)!} = \sum_{n=1}^{\infty} \frac{(-)^{n+1} 2n x^{2n-1}}{(2n-1)!}$$

$$= 4 \sum_{n=1}^{\infty} \frac{(-)^{n+1} n^2 x^{2n-1}}{(2n)!}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-)^{n+1} n^2}{(2n)!} = \frac{1}{4} \frac{d}{dx} [x \sin x] \Big|_{x=1}$$

$$= \frac{1}{4} (\cos(1) + \sin(1)) = 0.3454$$

$$\text{b)} \left( dx \sum_{n=1}^{\infty} \frac{(-)^{n+1} x^{2n}}{(2n-1)!} \right) = \sum_{n=1}^{\infty} \frac{(-)^{n+1} x^{2n+1}}{(2n-1)(2n-1)!}$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-)^{n+1} n x^{2n+1}}{(2n+1)!}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-)^{n+1} n}{(2n+1)!} = \frac{1}{2} \left[ dx x \sin x \right] \Big|_{x=1}$$

$$= \frac{1}{2} \left. \sin x - x \cos x \right|_{x=1}$$

$$= \frac{1}{2} (\sin(1) - \cos(1)) = 0.1566$$

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$$c) \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{(2n-1)!} \frac{\pi^{2n}}{4^n}$$

$$= 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{(2n)!} \left(\frac{\pi}{2}\right)^{2n}$$

$$= \pi \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{2n!} \left(\frac{\pi}{2}\right)^{2n-1}$$

$$= \frac{\pi}{4} \left. \frac{d}{dx} x \sin x \right|_{x=\pi/2}$$

$$= \frac{\pi}{4} \cdot \sin x + x \cos x \Big|_{x=\pi/2}$$

$$= \frac{\pi}{4} \approx 0.7854$$

$$d) \frac{d^2}{dx^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n-1)!}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n(2n-1)x^{2n-2}}{(2n-1)!}$$

$$\stackrel{n \rightarrow n+1}{=} \sum_{n=0}^{\infty} (-1)^n \frac{2(n+1)2n+1x^{2n}}{(2n+1)!}$$

$$= 2 \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)x^{2n}}{2n!}$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)}{2n!} = \left. \frac{d^2}{dx^2} \cos x - x \sin x \right|_{x=1} = \frac{2 \cos 1 - \sin 1}{2} \Big|_{x=1}$$

$$= \underline{\underline{\frac{2 \cos 1 - \sin 1}{2}}} = 0.1196$$

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### Additional Problem 1

$$a) \Gamma(m)\Gamma(n) = \int_0^\infty dt \int_0^\infty ds e^{-t} t^{m-1} e^{-s} s^{n-1}$$

$$t=ux \quad s=u(1-x) \quad \int_0^\infty dt \int_0^\infty ds = \int_0^\infty du u \int_0^1 dx$$

$$= \int_0^\infty du u^{m+n-1} e^{-u} \int_0^1 dx x^{m-1}(1-x)^{n-1}$$

$$= \Gamma(m+n) B(m, n)$$

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$b) \quad B(m, m) = \int_0^1 dx [x(1-x)]^{m-1} \quad x = \frac{1-u}{2} \quad -1 < u < 1$$

$$= \int_{-1}^1 \frac{du}{2} \left[ \frac{1-u^2}{4} \right]^{m-1}$$

$$= \frac{1}{2^{2m-1}} \int_{-1}^1 du (1-u^2)^{m-1}$$

$$= \frac{1}{2^{2m-1}} 2 \int_0^1 du (1-u^2)^{m-1} \quad y = u^2$$

$$= \frac{1}{2^{2m-1}} \int_0^1 dy y^{-\frac{1}{2}} (1-y)^{m-1}$$

$$= \frac{1}{2^{2m-1}} \frac{\Gamma(\frac{1}{2})\Gamma(m)}{\Gamma(m+\frac{1}{2})} = \frac{\sqrt{\pi}}{2^{2m-1}} \frac{\Gamma(m)}{\Gamma(m+\frac{1}{2})}$$

$$\Rightarrow \Gamma(2m) = \frac{2^{2m-1}}{\sqrt{\pi}} \Gamma(m)\Gamma(m+\frac{1}{2})$$

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### Additional Problem

$$\begin{aligned}
 a) n! &= \int_0^\infty dt e^{-t} t^n \\
 &= \int_0^\infty dt e^{-n\ln t - t} \\
 &= n^{n+1} \int_0^\infty ds e^{-n(\ln s - s)} \\
 &\approx n^{n+1} \int_0^\infty ds e^{-n} e^{-\frac{n}{2}(s-1)^2} \\
 &\approx n^{n+1} e^{-n} \sqrt{\frac{2\pi}{n}} \\
 &\approx \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} + o(n)
 \end{aligned}$$

$f(s) = \ln s - s$   
 $f'(s) = \frac{1}{s} - 1$   
 $f''(s) = -\frac{1}{s^2}$   
 $f(s) = -\frac{1}{2} - \frac{(s-1)^2}{2}$

$$\ln n! \approx (n-\frac{1}{2}) \ln n - n + \frac{1}{2} \ln 2\pi + o(n)$$

### b) Euler MacLaurin

$$\int_1^n dx \ln x = x \ln x - x \Big|_1^n = n \ln n - n + 1$$

$$\begin{aligned}
 &= \frac{1}{2} f(1) + f(2) + \dots + f(n-1) + \frac{1}{2} f(n) \\
 &+ \sum_{k=2}^m \frac{B_k}{k!} [f^{(k-1)}(n) - f^{(k-1)}(1)] + R_{m,n}
 \end{aligned}$$

$$\begin{aligned}
 f(x) &= \ln x \\
 f'(x) &= \frac{1}{x} \\
 f''(x) &= -\frac{1}{x^2} \\
 f'''(x) &= \frac{2}{x^3} \\
 f^{(4)}(x) &\sim -3! \frac{1}{x^4} \\
 &\vdots \\
 f^{(k-1)}(x) &= (-1)^{k+1} \frac{(k-2)!}{x^{k-1}}
 \end{aligned}$$

$$= \ln n! - \frac{1}{2} \ln n + \sum_{k=2}^{\infty} \frac{B_k}{k(k-1)} \left[ \frac{1}{n^{k-1}} - 1 \right] + R_{m,n}$$

where  $R_{m,n} = o\left(\frac{1}{n^m}\right)$

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rearranging we get

$$\ln n! = \left(n + \frac{1}{2}\right) \ln n - n + 1 - \sum_{k=2}^m \frac{B_k (-)^k}{k(k-1)} + \sum_{k=2}^m \frac{B_k (-)^k}{k(k-1)n^{k-1}} + R_{m,n}$$

to get correct asymptotic series replace

$$1 - \sum_{k=2}^m \frac{B_k (-)^k}{k(k-1)} \rightarrow \frac{1}{2} \ln 2\pi$$

$$\boxed{\ln n! = \left(n + \frac{1}{2}\right) \ln n - n + \frac{1}{2} \ln 2\pi + \sum_{k=2}^m \frac{B_k (-)^k}{k(k-1)} \frac{1}{n^{k-1}} + R_{m,n}}$$