

(1)

HW 1 Solutions

Cahill [1.13] $\tilde{f} = f - \frac{(f, g)}{(g, g)} g$

$$(\tilde{f}, \tilde{f}) \geq 0 \Rightarrow (f, f) - 2 \frac{(f, g)(g, f)}{(g, g)} + \frac{(g, f)(g, f)}{(g, g)^2} \geq 0$$

$$\cancel{(f, f) - \frac{|(f, g)|^2}{(g, g)}} \geq 0$$

$$\Rightarrow \underline{(f, g) = 0}$$

[1.18] $s_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ already normalized

$$\hat{s}_2 = s_2 - (\hat{s}_1, s_2) \hat{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{s}_3 = s_3 - (\hat{s}_1, s_3) \hat{s}_1 - (\hat{s}_2, s_3) \hat{s}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

[1.19] $s_1' \rightarrow \hat{s}_1' = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\hat{s}_2' = s_2' - (\hat{s}_1', s_2') \hat{s}_1' = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \rightarrow \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$\hat{s}_3' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Different basis

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$$1.28 \quad (\vec{\theta} \cdot \vec{\sigma})^2 = |\vec{\theta}|^2 \mathbb{1}$$

$$\begin{aligned} \exp\left(-i \frac{\vec{\theta} \cdot \vec{\sigma}}{2}\right) &= \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{-i}{2}\right)^n (\vec{\theta} \cdot \vec{\sigma})^n \\ &= \mathbb{1} \sum_{\text{even } n} \frac{1}{n!} \left(\frac{-i}{2}\right)^n \theta^{2n} + \vec{\theta} \cdot \vec{\sigma} \sum_{\text{odd } n} \frac{1}{n!} \left(\frac{i}{2}\right)^n \theta^n \\ &= \mathbb{1} \cos\left(\frac{\theta}{2}\right) - i \vec{\theta} \cdot \vec{\sigma} \sin\left(\frac{\theta}{2}\right) \end{aligned}$$

$$1.29 \quad -i J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad -i J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad -i J_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow -i \vec{\theta} \cdot \vec{J} = \begin{pmatrix} 0 & \theta_3 - \theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{pmatrix}$$

$$\begin{aligned} \text{Characteristic Eqn.} \quad &\begin{vmatrix} -\lambda & \theta_3 - \theta_2 \\ -\theta_3 & -\lambda & \theta_1 \\ \theta_2 & -\theta_1 & -\lambda \end{vmatrix} = -\lambda^3 - \lambda (\theta_1^2 + \theta_2^2 + \theta_3^2) \\ &= -\lambda^3 - \lambda \vec{\theta}^2 \end{aligned}$$

eigenvalues are $0, \pm i|\vec{\theta}|$

$$1.30 \quad \sum_i \lambda_i = \text{Tr}[M] \quad \text{where } \lambda_i \text{ are eigenvalues of } M$$

$$\text{if } M_{ij} = -M_{ji}, \quad \text{Tr}[M] = \sum_i M_{ii} = 0$$

$$\Rightarrow \sum_i \lambda_i = 0$$

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1.31] $-i\vec{\theta} \cdot \vec{J}$ has characteristic equation
 $-\lambda^3 - |\vec{\theta}|^2 = 0$

$$\Rightarrow (-i\vec{\theta} \cdot \vec{J})^3 = -|\vec{\theta}|^2 i\vec{\theta} \cdot \vec{J} = -|\vec{\theta}|^2 (-i\vec{\theta} \cdot \vec{J})$$

$$\Rightarrow (-i\vec{\theta} \cdot \vec{J})^{2n+1} = (-)^n |\vec{\theta}|^n - i\vec{\theta} \cdot \vec{J}$$

$$(-i\vec{\theta} \cdot \vec{J})_a^2 = -\theta_i \theta_j \epsilon_{ijk} \epsilon_{jbc}$$

$$= -\theta_i \theta_j \epsilon_{ijk} \epsilon_{jbc}$$

$$= -\theta_i \theta_j (\delta_{ac} - \delta_{ab} \delta_{jc})$$

$$= -|\vec{\theta}|^2 (\delta_{ac} - \theta_a \theta_c)$$

$$\Rightarrow (-i\vec{\theta} \cdot \vec{J})^2 = -|\vec{\theta}|^2 (\mathbb{I} - \vec{\theta} \vec{\theta}^T)$$

$$(-i\vec{\theta} \cdot \vec{J})^4 = -|\vec{\theta}|^2 (-i\vec{\theta} \cdot \vec{J})^2 \text{ etc}$$

$$e^{-i\vec{\theta} \cdot \vec{J}} = \mathbb{I} - i\vec{\theta} \cdot \vec{J} + \frac{(-i\vec{\theta} \cdot \vec{J})^2}{2} + \frac{(-i\vec{\theta} \cdot \vec{J})^3}{3!} + \frac{(-i\vec{\theta} \cdot \vec{J})^4}{4!}$$

$$= \mathbb{I} \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \dots \right) - i\vec{\theta} \cdot \vec{J} \left(\theta - \frac{\theta^3}{3!} + \dots \right)$$

$$+ \vec{\theta} \cdot \vec{\theta}^T \left(\frac{\theta^2}{2} - \frac{\theta^4}{4!} + \dots \right)$$

$$\boxed{e^{-i\vec{\theta} \cdot \vec{J}} = \mathbb{I} \cos \theta - i\vec{\theta} \cdot \vec{J} \sin \theta + (1 - \cos \theta) \vec{\theta} \vec{\theta}^T}$$

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1.32

In[42]:= **A** = {{1, 2, 3}, {3, 0, -1}}

Out[42]= {{1, 2, 3}, {3, 0, -1}}

In[43]:= **MatrixForm[A]**

Out[43]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \end{pmatrix}$$

In[44]:= **SingularValueDecomposition[A]**

Out[44]= {{{{1, 0}, {0, 1}}, {{Sqrt[14], 0, 0}, {0, Sqrt[10], 0}}}, {{{1/Sqrt[14], 3/Sqrt[10], 1/Sqrt[35]}, {Sqrt[2/7], 0, -Sqrt[5/7]}, {3/Sqrt[14], -1/Sqrt[10], 3/Sqrt[35]}}}}

In[45]:= **U** = {{1, 0}, {0, 1}}

Out[45]= {{1, 0}, {0, 1}}

In[46]:= **Diag** = {{Sqrt[14], 0, 0}, {0, Sqrt[10], 0}}

Out[46]= {{Sqrt[14], 0, 0}, {0, Sqrt[10], 0}}

In[47]:= **V** = {{1/Sqrt[14], 3/Sqrt[10], 1/Sqrt[35]}, {Sqrt[2/7], 0, -Sqrt[5/7]}, {3/Sqrt[14], -1/Sqrt[10], 3/Sqrt[35]}}

Out[47]= {{1/Sqrt[14], 3/Sqrt[10], 1/Sqrt[35]}, {Sqrt[2/7], 0, -Sqrt[5/7]}, {3/Sqrt[14], -1/Sqrt[10], 3/Sqrt[35]}}

In[48]:= **MatrixForm[U.Diag.Transpose[V]]**

Out[48]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \end{pmatrix}$$

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1.34 $\epsilon_{ijk}\epsilon_{imn}$ clearly must have $j \neq k$ with
then

$$\begin{aligned} \epsilon_{ijk}\epsilon_{imn} &= +1 && \text{if } i=m \ n=k \\ &= -1 && \text{if } j=n \ k=m \end{aligned}$$

$$\text{e.g. } \epsilon_{112}\epsilon_{112} = \overset{0}{\cancel{\epsilon_{112}}}^0 + \overset{0}{\cancel{\epsilon_{212}}}^0 + \overset{0}{\cancel{\epsilon_{312}}}^0 = 1$$

$$\epsilon_{112}\epsilon_{221} = \underset{0}{\cancel{\epsilon_{112}}}\epsilon_{121} + \underset{0}{\cancel{\epsilon_{212}}}\epsilon_{221} + \underset{+1}{\cancel{\epsilon_{312}}}\epsilon_{321} = -1$$

$$\Rightarrow \underline{\epsilon_{ijk}\epsilon_{imn} = \delta_{im}\delta_{kn} - \delta_{jn}\delta_{km}}$$

$$\text{RHB 8.61 a) } \vec{e}_i \cdot \vec{f}_j = \vec{e}_i \cdot H_{jk} \vec{e}_k = H_{jk} \vec{e}_i \cdot \vec{e}_k$$

$$= H_{jk} G_{ik}$$

$$= H_{jk} G_{ki} \quad G \text{ is symmetric}$$

$$= S_{ij} \quad H = G^{-1}$$

$$\vec{f}_i \cdot \vec{f}_j = H_{ia} \vec{e}_a \cdot H_{jb} \vec{e}_b = H_{ia} G_{ab} H_{jb}$$

$$= H_{ia} G_{ab} H_{bj}$$

$$= (H \times G)_{ij}$$

$$= H_{ij}$$

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b) $\vec{u} = u_i e_i$; $\vec{v} = v_j f_j$

$$|\vec{u}| = \sqrt{u_i g_{ij} u_j} \quad \vec{u} \cdot \vec{v} = u_i v_j$$

$$|\vec{v}| = \sqrt{v_j H_{ij} v_i}$$

c) $\cos \pi_3 = \frac{1}{2} =$

$$\vec{G} = a \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \quad H = \frac{1}{a} \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

d) let P be plane containing $\vec{v}_1, \vec{v}_2, \vec{v}_3$
 $(v_1 = \frac{1}{2}\hat{e}_1, v_2 = \frac{1}{2}\hat{e}_2, v_3 = \frac{1}{2}\hat{e}_3)$

normal to P is $\alpha (\vec{v}_1 - \vec{v}_2) \times (\vec{v}_2 - \vec{v}_3) = \vec{v}_1 \times \vec{v}_2 + \vec{v}_2 \times \vec{v}_3 + \vec{v}_3 \times \vec{v}_1$

unit normal is $\hat{n} = \frac{\vec{v}_1 \times \vec{v}_2 + \vec{v}_2 \times \vec{v}_3 + \vec{v}_3 \times \vec{v}_1}{|\vec{v}_1 \times \vec{v}_2 + \vec{v}_2 \times \vec{v}_3 + \vec{v}_3 \times \vec{v}_1|}$

ii) $\theta = \cos^{-1} \frac{\vec{v}_1 \cdot \hat{n}}{|\vec{v}_1|} = \cos^{-1} \left(\frac{\vec{v}_1 \cdot \vec{v}_2 \times \vec{v}_3}{|\vec{v}_1 \times \vec{v}_2 + \vec{v}_2 \times \vec{v}_3 + \vec{v}_3 \times \vec{v}_1| |\vec{v}_1|} \right)$

i) $D = \text{dist from } O \text{ to plane } P$

$$= \hat{n} \cdot \vec{v}_1 = |\vec{v}_1| \cos \theta$$

$$= \frac{\vec{v}_1 \cdot \vec{v}_2 \times \vec{v}_3}{|\vec{v}_1 \times \vec{v}_2 + \vec{v}_2 \times \vec{v}_3 + \vec{v}_3 \times \vec{v}_1|}$$

(7)

- 8.7) a) $(U^{-1}AU)^T = U^T A^T (U^{-1})^T = U^{-1}AU \quad \checkmark$
- b) $(iA)^T = -iA^T = -i(-A) = iA$
- c) $(AB)^T = B^T A^T = BA = AB \text{ iff } [A, B] = 0$
- d) $A = (I-S)(I+S)^{-1}$

$$(I+S)A = I - S \quad \leftarrow \text{apply transpose}$$

$$A^T(I+S^T) = I - S^T$$

$$A^T(I-S) = I + S$$

$$A^T = (I+S)(I+S)^{-1}$$

$$\begin{aligned} AA^T &= (I+S)(I+S)^{-1}(I-S)(I+S)^{-1} \\ &= (I+S)(I+S)^{-1} \end{aligned}$$

$$AA^T = I$$

$$(I+S)A = I - S$$

$$S(I+A) = I - A$$

$$S = (I-A)(I+A)^{-1} \quad C_\theta = \cos \theta \quad S_\theta = \sin \theta$$

$$I - A = \begin{pmatrix} 1 - C_\theta & -S_\theta \\ S_\theta & 1 - C_\theta \end{pmatrix} \quad I + A = \begin{pmatrix} 1 + C_\theta & S_\theta \\ -S_\theta & 1 + C_\theta \end{pmatrix}$$

$$(I+A)^{-1} = \frac{1}{2(1+C_\theta)} \begin{pmatrix} 1+C_\theta & -S_\theta \\ S_\theta & 1+C_\theta \end{pmatrix}$$

$$(I-A)(I+A)^{-1} = \frac{1}{2(1+C_\theta)} \begin{pmatrix} 1-C_\theta & -S_\theta \\ S_\theta & 1-C_\theta \end{pmatrix} \begin{pmatrix} 1+C_\theta & -S_\theta \\ S_\theta & 1+C_\theta \end{pmatrix}$$

$$= \frac{1}{2(1+C_\theta)} \begin{pmatrix} 0 & -2S_\theta \\ +2S_\theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{S_\theta}{1+C_\theta} \\ \frac{S_\theta}{1+C_\theta} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \tan \theta/2 \\ -\tan \theta/2 \end{pmatrix}$$

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$$c) v = (1+k)(1-k)^{-1}$$

$(1-k)v = 1+k \leftarrow$ apply hermitian conjugate

$$v^T - V^T k^T = 1 + k^T$$

$$V^T(1+k) = 1-k$$

$$V^T = (1-k)(1+k)^{-1}$$

$$V^T v = (1-k)(1+k)^{-1}(1+k)(1-k)^{-1} = \mathbb{I}$$

8.83 $(AB)^T + B^{-1}A = 0$

$$B^T A^T + B^{-1}A = 0$$

a) $B^T = B^{-1}$ orthogonal

$$\Rightarrow B^T(A^T + A) = 0$$

$$\Rightarrow A^T = -A$$

b) $B B^T A^T + A = 0$

$$B B^T A^T = -A$$

$$\text{Det}[B B^T A^T] = -\text{Det}A$$

$$\Rightarrow (\text{Det } B)^2 \text{ Det } A = -\text{Det } A$$

solutions \Rightarrow a) $(\text{Det } B)^2 = -1$

b) $\text{Det } A = 0 \quad \text{Det } B = \text{anything}$

but B is real matrix $\text{Det } B^2 > 0$ so only solution is b).

(9)

8.10] S_x, S_y, S_z are Pauli matrices

$$S_i S_j = \delta_{ij} + i \epsilon_{ijk} S_k$$

$$(S_i^2 = 1 \quad [S_i, S_j] = i \epsilon_{ijk} S_k)$$

$$\Rightarrow a_i S_i \cdot b_j S_j = a_i b_j (\delta_{ij} + i \epsilon_{ijk} S_k) \\ = \vec{a} \cdot \vec{b} + i \vec{S} \cdot (\vec{a} \times \vec{b}) \quad \checkmark$$

$$[\vec{S} \cdot \vec{a}, \vec{S} \cdot \vec{b}] = 2i \vec{S} \cdot (\vec{a} \times \vec{b}) \\ \Leftarrow 0 \text{ iff } \vec{a} \parallel \vec{b}$$

Q.12] $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ has eigenvectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ev = 1

$$|\psi_1\rangle \begin{pmatrix} 1 \\ -\sqrt{\alpha} \\ 0 \end{pmatrix} \text{ ev} = 1 - \sqrt{\alpha}$$

$$|\psi_2\rangle \begin{pmatrix} 1 \\ \sqrt{\alpha} \\ 0 \end{pmatrix} \text{ ev} = 1 + \sqrt{\alpha}$$

$$\langle e_2 | e_1 \rangle \propto (1 + \sqrt{\alpha})^* \begin{pmatrix} 1 \\ -\sqrt{\alpha} \\ 0 \end{pmatrix} = 1 - \frac{|\alpha|}{|\beta|}$$

(10)

i) orthonormality $\Rightarrow |\alpha| = |\beta|$ (we will get the same condition
if we require the matrix be normal)ii) real eigenvalue if $\sqrt{\alpha\beta}$ is real.

$$\text{if } \alpha = e^{i\theta}|\alpha|$$

$$\beta = e^{-i\theta}|\beta|$$

$$i) |\kappa| = |\beta| \quad \sqrt{\alpha\beta} = e^{i(\theta-\phi)/2} \sqrt{|\alpha||\beta|}$$

$\Rightarrow \theta = \phi$ so $\beta = \alpha^*$ and matrix
is Hermitian

$$8.14) \quad u = A + iB$$

$$a) u^+ = (A + iB)(A - iB) = A^2 + B^2 + i[A, B]$$

$$u^+u = (A - iB)(A + iB) = A^2 - B^2 + i[A, B]$$

$$\text{for } uu^+ = u^+u = \mathbb{1} \quad [A, B] = 0 \quad A^2 + B^2 = 1$$

c) Assume non degeneracy of A's evals

$$A|v\rangle = \alpha|v\rangle$$

$$AB|v\rangle = BA|v\rangle = \alpha B|v\rangle$$

$\Rightarrow B|v\rangle$ is eigenvalue of A w/ evaln a
non degeneracy $\Rightarrow B|v\rangle \propto |v\rangle$

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if $B|v\rangle \times |v\rangle \Rightarrow B|v\rangle = b|v\rangle$ for some b ie $|v\rangle$ is evector of B also.

a) let $|v\rangle$ be evector of A, B

$$A^2 + B^2 |v\rangle = |v\rangle = a^2 + b^2 |v\rangle$$

$$\begin{aligned} |v\rangle &= A+iB|v\rangle \\ &= a+ib|v\rangle \end{aligned}$$

since $a^2+b^2=1$ $a+ib = e^{i\theta}$ for some θ

$$\Rightarrow |v\rangle = e^{i\theta}|v\rangle$$