Related books by this author include *Contemporary Crystallography*. New York: McGraw-Hill (1970), *Crystal Structure Analysis*, Krieger (1979) (reprint, 1960), and *Introduction to Crystal Geometry*, Krieger (1977) (reprint, 1971).

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A collection of reprints of significant papers on **SU**(3) and the particles of high-energy physics. The several introductory sections by Gell-Mann and Ne'eman are especially helpful.

HAMERMESH, M., *Group Theory and Its Application to Physical Problems*. Reading, Mass.: Addison-Wesley (1962).
A detailed, rigorous account of both finite and continuous groups. The 32 point groups are developed. The continuous groups are treated with Lie algebra included. A wealth of applications to atomic and nuclear physics.

HIGMAN, B., *Applied Group-Theoretic and Matrix Methods*. New York: Dover (1964), Oxford: Oxford University Press (1955).
A rather complete and unusually intelligible development of matrix analysis and group theory.

PARK, D., "Resource Letter SP-1 on Symmetry in Physics." *Am. J. Phys.* **36**, 577–584 (1968).
Includes a large selection of basic references on group theory and its applications to physics: atoms, molecules, nuclei, solids, and elementary particles.

RAM, B., *Am. J. Phys.* **35**, 16 (1967).
An excellent discussion of the application of **SU**(3) to the strongly interacting particles (baryons). For a sequel to this see R. D. YOUNG, "Physics of the Quark Model." *Am. J. Phys.* **41**, 472 (1973).

ROSE, M. E., *Elementary Theory of Angular Momentum*. New York: Wiley (1957).
As part of the development of the quantum theory of angular momentum, Rose includes a detailed and readable account of the rotation group.

WIGNER, E. P., *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra*. Translated by J. J. Griffin. New York and London: Academic Press (1959).
This is the classic reference on group theory for the physicist. The rotation group is treated in considerable detail. There are a wealth of applications to atomic physics.

5 INFINITE SERIES

5.1 FUNDAMENTAL CONCEPTS

Infinite series, literally summations of an infinite number of terms, occur frequently in both pure and applied mathematics. They may be used by the pure mathematician to define functions as a fundamental approach to the theory of functions, as well as for calculating accurate values of transcendental constants and transcendental functions. In the mathematics of science and engineering infinite series are ubiquitous, for they appear in the evaluation of integrals (Section 5.6 and 5.7), in the solution of differential equations (Sections 8.5 and 8.6), and as Fourier series (Chapter 14) and compete with integral representations for the description of a host of special functions (Chapters 11, 12, and 13). In Section 16.3 the Neumann series solution for integral equations provides one more example of the occurrence and use of infinite series.

Right at the start we face the problem of attaching meaning to the sum of an infinite number of terms. The usual approach is by partial sums. If we have a sequence of infinite terms $u_1, u_2, u_3, u_4, u_5, \ldots$, we define the ith partial sum as

$$s_i = \sum_{n=1}^{i} u_n. \tag{5.1}$$

This is a finite summation and offers no difficulties. If the partial sums s_i converge to a (finite) limit as $i \to \infty$,

$$\lim_{i\to\infty} s_i = S, \tag{5.2}$$

the infinite series $\sum_{n=1}^{\infty} u_n$ is said to be *convergent* and to have the value S. Note carefully that we reasonably, plausibly, but still arbitrarily *define* the infinite series as equal to S. The reader should also note that a necessary condition for this convergence to a limit is that $\lim_{n\to\infty} u_n = 0$. This condition, however, is *not* sufficient to guarantee convergence. Equation 5.2 is usually written in formal mathematical notation:

> The condition for the existence of a limit S is that for each $\varepsilon > 0$, there is a fixed N such that
>
> $$|S - s_i| < \varepsilon, \qquad \text{for } i > N.$$

This condition is often derived from the *Cauchy criterion* applied to the partial sums s_i. The Cauchy criterion is:

> A necessary and sufficient condition that a sequence (s_i) converge

is that for each $\varepsilon > 0$ there is a fixed number N such that

$$|s_j - s_i| < \varepsilon \quad \text{for all } i, j > N.$$

This means that the individual partial sums must cluster together as we move far out in the sequence.

The Cauchy criterion may easily be extended to sequences of functions. We see it in this form in Section 5.5 in the definition of uniform convergence and in Section 9.4 in the development of Hilbert space.

Our partial sums s_i may not converge to a single limit but may oscillate, as in the case

$$\sum_{n=1}^{\infty} u_n = 1 - 1 + 1 - 1 + 1 + \cdots - (-1)^n + \cdots. \tag{5.3}$$

Clearly, $s_i = 1$ for i odd but 0 for i even. There is no convergence to a limit, and series such as this one are labeled oscillatory.

For the series

$$1 + 2 + 3 + \cdots + n + \cdots \tag{5.4}$$

we have

$$s_n = \frac{n(n+1)}{2}. \tag{5.5}$$

As $n \to \infty$,

$$\lim_{n\to\infty} s_n = \infty. \tag{5.6}$$

Whenever the sequence of partial sums diverges (approaches $\pm\infty$), the infinite series is said to *diverge*. Often the term divergent is extended to include oscillatory series as well.

Because we evaluate the partial sums by ordinary arithmetic, the convergent series, defined in terms of a limit of the partial sums, assume a position of supreme importance. Two examples may clarify the nature of convergence or divergence of a series and will also serve as a basis for a further detailed investigation in the next section.

EXAMPLE 5.1.1 The Geometric Series

The geometrical sequence, starting with a and with a ratio r ($r \geq 0$), is given by

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots.$$

The nth partial sum is given by[1]

$$s_n = a\frac{1 - r^n}{1 - r}. \tag{5.7}$$

[1] Multiply and divide $s_n = \sum_{m=0}^{n-1} ar^m$ by $1 - r$.

Taking the limit as $n \to \infty$,

$$\lim_{n\to\infty} s_n = \frac{a}{1 - r}, \qquad \text{for } r < 1. \tag{5.8}$$

Hence, by definition, the infinite geometric series converges for $r < 1$ and is given by

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}. \tag{5.9}$$

On the other hand, if $r \geq 1$, the necessary condition $u_n \to 0$ is not satisfied and the infinite series diverges.

EXAMPLE 5.1.2 The Harmonic Series

As a second and more involved example, we consider the harmonic series

$$\sum_{n=1}^{\infty} n^{-1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots. \tag{5.10}$$

We have the $\lim_{n\to\infty} u_n = \lim_{n\to\infty} 1/n = 0$, but this is not sufficient to guarantee convergence. If we group the terms (no change in order) as

$$1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + (\tfrac{1}{9} + \cdots + \tfrac{1}{16}) + \cdots, \tag{5.11}$$

it will be seen that each pair of parentheses encloses p terms of the form

$$\frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{p+p} > \frac{p}{2p} = \frac{1}{2}. \tag{5.12}$$

Forming partial sums by adding the parenthetical groups one by one, we obtain

$$\begin{aligned} s_1 &= 1, & S_8\ \cancel{s_4} &> \frac{5}{2}, \\ s_2 &= \frac{3}{2}, & S_{16}\ \cancel{s_5} &> \frac{6}{2}, \\ S_4\ \cancel{s_3} &> \frac{4}{2}, & s_n &> \frac{n+1}{2}. \end{aligned} \tag{5.13}$$

The harmonic series considered in this way is certainly divergent.[2] An alternate and independent demonstration of its divergence appears in Section 5.2.

Using the binomial theorem[3] (Section 5.6), we may expand the function $(1 + x)^{-1}$:

[2] The (finite) harmonic series appears in an interesting note on the maximum stable displacement of a stack of coins, Johnson, P. R., "The Leaning Tower of Lire." *Am. J. Phys.* **23**, 240 (1955).

[3] Actually Eq. 5.14 may be taken as an identity and verified by multiplying both sides by $1 + x$.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-x)^{n-1} + \cdots. \tag{5.14}$$

If we let $x \to 1$, this series becomes

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots, \tag{5.15}$$

a series that we labeled oscillatory earlier in this section. Although it does not converge in the usual sense, meaning can be attached to this series. Euler, for example, assigned a value of $\frac{1}{2}$ to this oscillatory sequence on the basis of the correspondence between this series and the well-defined function $(1+x)^{-1}$. Unfortunately, such correspondence between series and function is not unique and this approach must be refined. Other methods of assigning a meaning to a divergent or oscillatory series, methods of defining a sum, have been developed. In general, however, this aspect of infinite series is of relatively little interest to the scientist or the engineer. An exception to this statement, the very important asymptotic or semiconvergent series, is considered in Section 5.10.

EXERCISES

5.1.1 Show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}.$$

Hint. Show (by mathematical induction) that $s_m = m/(2m+1)$.

5.1.2 Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Find the partial sum s_m and verify its correctness by mathematical induction.
Note. The method of expansion in partial fractions, Section 15.8, offers an alternative way of solving Exercises 5.1.1 and 5.1.2.

5.2 CONVERGENCE TESTS

Although nonconvergent series may be useful in certain special cases, (compare Section 5.10), we usually insist, as a matter of convenience if not necessity, that our series be convergent. It therefore becomes a matter of extreme importance to be able to tell whether a given series is convergent. We shall develop a number of possible tests, starting with the simple and relatively insensitive tests and working up to the more complicated but quite sensitive tests.

For the present let us consider a series of positive terms, $a_n > 0$, postponing negative terms until the next section.

Comparison Test

If term by term a series of terms $u_n \le a_n$, in which the a_n form a convergent series, the series $\sum_n u_n$ is also convergent. Symbolically, we have

$$\sum_n a_n = a_1 + a_2 + a_3 + \cdots, \qquad \text{convergent},$$

$$\sum_n u_n = u_1 + u_2 + u_3 + \cdots.$$

If $u_n \le a_n$ for all n, then $\sum_n u_n \le \sum_n a_n$ and $\sum_n u_n$ therefore is convergent.

If term by term a series of terms $v_n \ge b_n$, in which the b_n form a divergent series, the series $\sum_n v_n$ is also divergent. Note that comparisons of u_n with b_n or v_n with a_n yield no information. Here we have

$$\sum_n b_n = b_1 + b_2 + b_3 + \cdots, \qquad \text{divergent},$$

$$\sum_n v_n = v_1 + v_2 + v_3 + \cdots.$$

If $v_n \ge b_n$ for all n, then $\sum_n v_n \ge \sum_n b_n$ and $\sum_n v_n$ therefore is divergent.

For the convergent series a_n we already have the geometric series, whereas the harmonic series will serve as the divergent series b_n. As other series are identified as either convergent or divergent, they may be used for the known series in this comparison test.

All tests developed in this section are essentially comparison tests. Figure 5.1 exhibits these tests and the interrelationships.

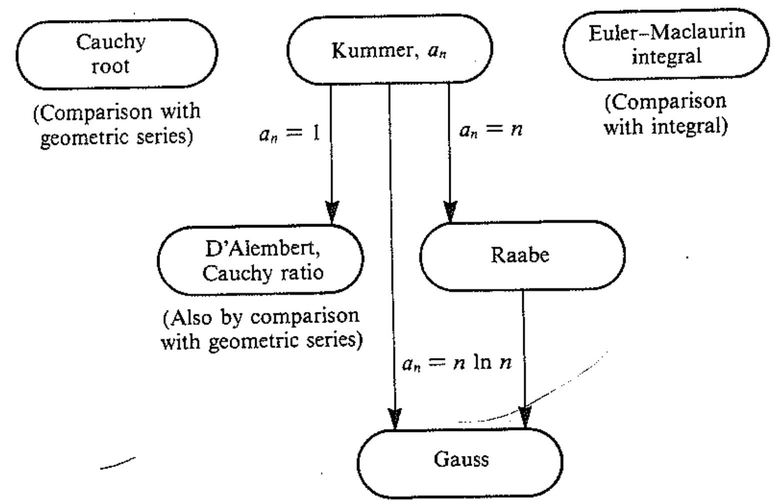


FIG. 5.1 Comparison tests

EXAMPLE 5.2.1 The p Series

Test $\sum_n n^{-p}$, $p = 0.999$, for convergence. Since $n^{-0.999} > n^{-1}$, and $b_n = n^{-1}$ forms the divergent harmonic series, the comparison test shows that $\sum_n n^{-0.999}$ is divergent. Generalizing, $\sum_n n^{-p}$ is seen to be divergent for all $p \le 1$.

Cauchy Root Test

If $(a_n)^{1/n} \le r < 1$ for all sufficiently large n, with r independent of n, then $\sum_n a_n$ is convergent. If $(a_n)^{1/n} \ge 1$ for all sufficiently large n, then $\sum_n a_n$ is divergent.

The first part of this test is verified easily by raising $(a_n)^{1/n} \le r$ to the nth power. We get

$$a_n \le r^n < 1.$$

Since r^n is just the nth term in a convergent geometric series, $\sum_n a_n$ is convergent by the comparison test. Conversely, if $(a_n)^{1/n} \ge 1$, then $a_n \ge 1$ and the series must diverge. This root test is particularly useful in establishing the properties of power series (Section 5.7).

D'Alembert or Cauchy Ratio Test

If $a_{n+1}/a_n \le r < 1$ for all sufficiently large n, and r is independent of n, then $\sum_n a_n$ is convergent. If $a_{n+1}/a_n \ge 1$ for all sufficiently large n, then $\sum_n a_n$ is divergent.

Convergence is proved by direct comparison with the geometric series $(1 + r + r^2 + \cdots)$. In the second part $a_{n+1} \ge a_n$ and divergence should be reasonably obvious. Although not quite so sensitive as the Cauchy root test, this D'Alembert ratio test is one of the easiest to apply and is widely used. An alternate statement of the ratio test is in the form of a limit:

If

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \begin{cases} < 1, & \text{convergence,} \\ > 1, & \text{divergence,} \\ = 1, & \text{indeterminant.} \end{cases} \tag{5.16}$$

Because of this final indeterminant possibility, the ratio test is likely to fail at crucial points, and more delicate, more sensitive tests are necessary.

The alert reader may wonder how this indeterminacy arose. Actually it was concealed in the first statement $a_{n+1}/a_n \le r < 1$. We might encounter $a_{n+1}/a_n < 1$ for all *finite* n but be unable to choose an $r < 1$ *and independent of* n such that $a_{n+1}/a_n \le r$ for all sufficiently large n. An example is provided by the harmonic series

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} < 1. \tag{5.17}$$

Since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1. \tag{5.18}$$

no fixed ratio $r < 1$ exists and the ratio test fails.

EXAMPLE 5.2.2 D'Alembert Ratio Test

Test $\sum_n n/2^n$ for convergence.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{1}{2} \cdot \frac{n+1}{n}. \tag{5.19}$$

Since

$$\frac{a_{n+1}}{a_n} \le \frac{3}{4} \qquad \text{for } n \ge 2, \tag{5.20}$$

we have convergence. Alternatively,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2} \tag{5.21}$$

and again—convergence.

Cauchy or Maclaurin Integral Test

This is another sort of comparison test in which we compare a series with an integral. Geometrically, we compare the area of a series of unit-width rectangles with the area under a curve.

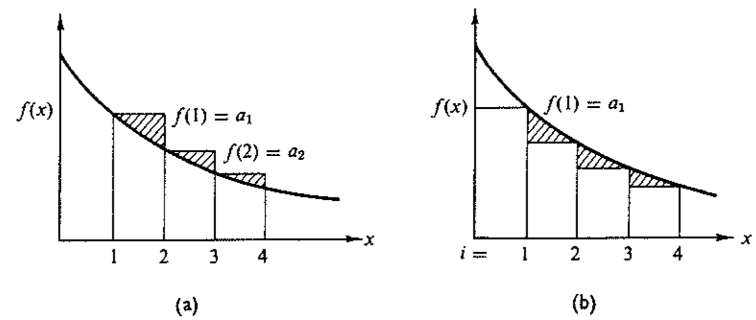


FIG. 5.2 (a) Comparison of integral and sum-blocks leading. (b) Comparison of integral and sum-blocks lagging

Let $f(x)$ be a continuous, monotonic decreasing function in which $f(n) = a_n$. Then $\sum_n a_n$ converges if $\int_1^\infty f(x)\,dx$ is finite and diverges if the integral is infinite. For the ith partial sum

$$s_i = \sum_{n=1}^{i} a_n = \sum_{n=1}^{i} f(n). \tag{5.22}$$

But

$$s_i > \int_1^{i+1} f(x)\,dx \tag{5.23}$$

by Fig. 5.2*a*, $f(x)$ being monotonic decreasing. On the other hand, from Fig. 5.2*b*,

$$s_i - a_1 < \int_1^{i} f(x)\,dx, \tag{5.24}$$

in which the series is represented by the inscribed rectangles. Taking the limit as $i \to \infty$, we have

$$\int_1^\infty f(x)\,dx < \sum_{n=1}^{\infty} a_n < \int_1^\infty f(x)\,dx + a_1. \tag{5.25}$$

Hence the infinite series converges or diverges as the corresponding integral converges or diverges.

This integral test is particularly useful in setting upper and lower bounds on the remainder of a series after some number of initial terms have been summed. That is,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n,$$

where

$$\int_{N+1}^{\infty} f(x)\,dx < \sum_{n=N+1}^{\infty} a_n < \int_{N+1}^{\infty} f(x)\,dx + a_{N+1}.$$

EXAMPLE 5.2.3 Riemann Zeta Function

The Riemann zeta function is defined by

$$\zeta(p) = \sum_{n=1}^{\infty} n^{-p}. \tag{5.26}$$

We may take $f(x) = x^{-p}$ and then

$$\begin{aligned} \int_1^{\infty} x^{-p}\,dx &= \left.\frac{x^{-p+1}}{-p+1}\right|_1^{\infty}, \qquad p \neq 1 \\ &= \ln x\Big|_1^{\infty}, \qquad p = 1. \end{aligned} \tag{5.27}$$

The integral and therefore the series are divergent for $p \leq 1$, convergent for $p > 1$. Hence Eq. 5.26 should carry the condition $p > 1$. This, incidentally, is an independent proof that the harmonic series ($p = 1$) diverges and diverges logarithmically. The sum of the first million terms $\sum^{1{,}000{,}000} n^{-1}$, is only 14.392 726. . . .

This integral comparison may also be used to set an upper limit to the Euler–Mascheroni constant[1] defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n} m^{-1} - \ln n\right). \tag{5.28}$$

Returning to partial sums,

$$s_n = \sum_{m=1}^{n} m^{-1} - \ln n < \int_1^n \frac{dx}{x} - \ln n + 1. \tag{5.29}$$

Evaluating the integral on the right, $s_n < 1$ for all n and therefore $\gamma < 1$. Exercise 5.2.12 leads to more restrictive bounds. Actually the Euler–Mascheroni constant is 0.577 215 66. . . .

[1] This is the notation of National Bureau of Standards, *Handbook of Mathematical Functions*. Applied Mathematics Series-55 (AMS-55).

Kummer's Test

This is the first of three tests that are somewhat more difficult to apply than the preceding tests. Their importance lies in their power and sensitivity. Frequently, at least one of the three will work when the simpler easier tests are indecisive. It must be remembered, however, that these tests, like those previously discussed, are ultimately based on comparisons. It can be shown that there is no most slowly converging convergent series and no most slowly diverging divergent series. This means that all convergence tests given here, including Kummer's, may fail sometime.

We consider a series of positive terms u_i and a sequence of finite positive constants a_i. If

$$a_n \frac{u_n}{u_{n+1}} - a_{n+1} \geq C > 0 \tag{5.30}$$

for all $n \geq N$, some fixed number,[2] then $\sum_{i=1}^{\infty} u_i$ converges. If

$$a_n \frac{u_n}{u_{n+1}} - a_{n+1} \leq 0 \tag{5.31}$$

and $\sum_{i=1}^{\infty} a_i^{-1}$ diverges, then $\sum_{i=1}^{\infty} u_i$ diverges.

The proof of this powerful test is remarkably simple. From Eq. 5.30, with C some positive constant,

$$\begin{aligned} Cu_{N+1} &\leq a_N u_N - a_{N+1}u_{N+1} \\ Cu_{N+2} &\leq a_{N+1}u_{N+1} - a_{N+2}u_{N+2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ Cu_n &\leq a_{n-1}u_{n-1} - a_n u_n \end{aligned} \tag{5.32}$$

Adding and dividing by C, ($C \neq 0$), we obtain

$$\sum_{i=N+1}^{n} u_i \leq \frac{a_N u_N}{C} - \frac{a_n u_n}{C}. \tag{5.33}$$

Hence for the partial sum, s_n,

$$\begin{aligned} s_n &\leq \sum_{i=1}^{N} u_i + \frac{a_N u_N}{C} - \frac{a_n u_n}{C} \\ &< \sum_{i=1}^{N} u_i + \frac{a_N u_N}{C}, \qquad \text{a constant, independent of } n. \end{aligned} \tag{5.34}$$

The partial sums therefore have an upper bound. With zero as an obvious lower bound, the series $\sum iu_i$ must converge.

Divergence is shown as follows. From Eq. 5.31

[2] With u_m finite, the partial sum s_N will always be finite for N finite. The convergence or divergence of a series depends on the behavior of the last infinity of terms, *not* on the first N terms.

$$a_n u_n \geq a_{n-1}u_{n-1} \geq \cdots \geq a_N u_N, \qquad n > N. \tag{5.35}$$

Thus

$$u_n \geq \frac{a_N u_N}{a_n} \tag{5.36}$$

and

$$\sum_{i=N+1}^{\infty} u_i \geq a_N u_N \sum_{i=N+1}^{\infty} a_i^{-1}. \tag{5.37}$$

If $\sum_{i=1}^{\infty} a_i^{-1}$ diverges, then by the comparison test $\sum_i u_i$ diverges.

Equations 5.30 and 5.31 are often given in a limit form:

$$\lim_{n\to\infty}\left(a_n \frac{u_n}{u_{n+1}} - a_{n+1}\right) = C. \tag{5.38}$$

Thus for $C > 0$ we have convergence, whereas for $C < 0$ (and $\sum_i a_i^{-1}$ divergent) we have divergence. It is perhaps useful to show the equivalence of Eq. 5.38 and Eqs. 5.30 and 5.31 and to show why indeterminacy creeps in when the limit $C = 0$. From the definition of limit

$$\left| a_n \frac{u_n}{u_{n+1}} - a_{n+1} - C \right| < \varepsilon \tag{5.39}$$

for all $n \geq N$ and all $\varepsilon > 0$, no matter how small ε may be. When the absolute value signs are removed,

$$C - \varepsilon < a_n \frac{u_n}{u_{n+1}} - a_{n+1} < C + \varepsilon. \tag{5.40}$$

Now if $C > 0$, Eq. 5.30 follows from ε sufficiently small. On the other hand, if $C < 0$, Eq. 5.31 follows. However, if $C = 0$, the center term $a_n(u_n/u_{n+1}) - a_{n+1}$ may be either positive or negative and the proof fails. The primary use of Kummer's test is to prove other tests such as Raabe's (compare also Exercise 5.2.3).

If the positive constants a_n of Kummer's test are chosen $a_n = n$, we have Raabe's test.

Raabe's Test

If $u_n > 0$ and if

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) \geq P > 1 \tag{5.41}$$

for all $n \geq N$, where N is a positive integer independent of n, then $\sum_i u_i$ converges. If

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) \leq 1, \tag{5.42}$$

then $\sum_i u_i$ diverges ($\sum n^{-1}$ diverges).

The limit form of Raabe's test is

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = P. \tag{5.43}$$

We have convergence for $P > 1$, divergence for $P < 1$, and no test for $P = 1$ exactly as with the Kummer test. This indeterminacy is pointed up by Exercise 5.2.4, which presents a convergent series and a divergent series with both series yielding $P = 1$ in Eq. 5.43.

Raabe's test is more sensitive than the d'Alembert ratio test because $\sum_{n=1}^{\infty} n^{-1}$ diverges more slowly than $\sum_{n=1}^{\infty} 1$. We obtain a still more sensitive test (and one that is relatively easy to apply) by choosing $a_n = n \ln n$. This is Gauss's test.

Gauss's Test

If $u_n > 0$ for all finite n and

$$\frac{u_n}{u_{n+1}} = 1 + \frac{h}{n} + \frac{B(n)}{n^2}, \tag{5.44a}$$

in which $B(n)$ is a bounded function of n for $n \to \infty$, then $\sum_i u_i$ converges for $h > 1$ and diverges for $h \leq 1$.

The ratio u_n/u_{n+1} of Eq. 5.44a often comes as the ratio of two quadratic forms:

$$\frac{u_n}{u_{n+1}} = \frac{n^2 + a_1 n + a_0}{n^2 + b_1 n + b_0}. \tag{5.44b}$$

It may be shown (Exercise 5.2.5) that we have convergence for $a_1 > b_1 + 1$ and divergence for $a_1 \leq b_1 + 1$.

The Gauss test is an extremely sensitive test of series convergence. It will work for all series the physicist is likely to encounter. For $h > 1$ or $h < 1$ the proof follows directly from Raabe's test

$$\begin{aligned}\lim_{n\to\infty} n\left[1 + \frac{h}{n} + \frac{B(n)}{n^2} - 1\right] &= \lim_{n\to\infty}\left[h + \frac{B(b)}{n}\right] \\ &= h.\end{aligned} \tag{5.45}$$

If $h = 1$, Raabe's test fails. However, if we return to Kummer's test and use $a_n = n \ln n$, Eq. 5.38 leads to

$$\begin{aligned}&\lim_{n\to\infty}\left\{n \ln n\left[1 + \frac{1}{n} + \frac{B(n)}{n^2}\right] - (n+1)\ln(n+1)\right\} \\ &= \lim_{n\to\infty}\left[n \ln n \cdot \frac{(n+1)}{n} - (n+1)\ln(n+1)\right] \\ &= \lim_{n\to\infty}(n+1)\left[\ln n - \ln n - \ln\left(1 + \frac{1}{n}\right)\right].\end{aligned} \tag{5.46}$$

Borrowing a result from Section 5.6 (which is not dependent on Gauss's test), we have

$$\lim_{n\to\infty} -(n+1)\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} -(n+1)\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\cdots\right) = -1 < 0. \tag{5.47}$$

Hence we have divergence for $h = 1$. This is an example of a successful application of Kummer's test in which Raabe's test had failed.

EXAMPLE 5.2.4 Legendre Series

The recurrence relation for the series solution of Legendre's equation (Section 8.5) may be put in the form

$$\frac{a_{2j+2}}{a_{2j}} = \frac{2j(2j+1)-l(l+1)}{(2j+1)(2j+2)}. \tag{5.48}$$

This is equivalent to u_{2j+2}/u_{2j} for $x = +1$. For $j \gg l$,[3]

$$\frac{u_{2j}}{u_{2j+2}} \to \frac{(2j+1)(2j+2)}{2j(2j+1)} = \frac{2j+2}{2j} = 1+\frac{1}{j}. \tag{5.49}$$

By Eq. 5.44*b* the series is divergent. Later we shall demand that the Legendre series be finite at $x = 1$. We shall eliminate the divergence by setting the parameter $n = 2j_0$, an even integer. This will truncate the series, converting the infinite series into a polynomial.

Improvement of Convergence

This section so far has been concerned with establishing convergence as an abstract mathematical property. In practice, the *rate* of convergence may be of considerable importance. Here we present one method of improving the rate of convergence of a convergent series. Other techniques are given in Sections 5.4 and 5.9.

The basic principle of this method, due to Kummer, is to form a linear combination of our slowly converging series and one or more series whose sum is known. For the known series the collection

$$\alpha_1 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\alpha_2 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$$

$$\alpha_3 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)} = \frac{1}{18}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\alpha_p = \sum_{n=1}^{\infty} \frac{1}{n(n+1)\cdots(n+p)} = \frac{1}{p\cdot p!}$$

[3] The n dependence enters $B(n)$ but does not affect h.

is particularly useful.[4] The series are combined term by term and the coefficients in the linear combination chosen to cancel the most slowly converging terms.

EXAMPLE 5.2.5 Riemann Zeta Function, $\zeta(3)$

Let the series to be summed be $\sum_{n=1}^{\infty} n^{-3}$. In Section 5.9 this is identified as a Riemann zeta function, $\zeta(3)$. We form a linear combination

$$\sum_{n=1}^{\infty} n^{-3} + a_2\alpha_2 = \sum_{n=1}^{\infty} n^{-3} + \frac{a_2}{4}.$$

α_1 is not included since it converges more slowly than $\zeta(3)$. Combining terms, we obtain on the left-hand side

$$\sum_{n=1}^{\infty} \left\{\frac{1}{n^3} + \frac{a_2}{n(n+1)(n+2)}\right\} = \sum_{n=1}^{\infty} \frac{n^2(1+a_2)+3n+2}{n^3(n+1)(n+2)}.$$

If we choose $a_2 = -1$, the preceding equations yield

$$\zeta(3) = \sum_{n=1}^{\infty} n^{-3} = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{3n+2}{n^3(n+1)(n+2)}.$$

The resulting series may not be beautiful but it does converge as n^{-4}, appreciably faster than n^{-3}. A more convenient form comes from Exercise 5.2.21. There, the symmetry leads to convergence as n^{-5}.

The method can be extended including $a_3\alpha_3$ to get convergence as n^{-5}, $a_n\alpha_n$ to get convergence as n^{-6}, and so on. Eventually, you have to reach a compromise between how much algebra you do and how much arithmetic the computing machine does. As computing machines get larger and faster, the balance is steadily shifting to less algebra for you and more arithmetic for the machine.

EXERCISES

5.2.1 (a) Prove that if

$$\lim_{n\to\infty} n^p u_n \to A < \infty; \qquad p > 1,$$

the series $\sum_{n=1}^{\infty} u_n$ converges.

(b) Prove that if

$$\lim_{n\to\infty} n u_n = A > 0,$$

the series diverges. (The test fails for $A = 0$.)

These two tests, known as limit tests, are often convenient for establishing the convergence or divergence of a series. They may be treated as comparison tests, comparing with

[4] These series sums may be verified by expanding the forms by partial fractions, writing out the initial terms and inspecting the pattern of cancellation of positive and negative terms.

$$\sum n^{-q}, \qquad 1 \le q < p.$$

5.2.2 If

$$\lim_{n\to\infty} \frac{b_n}{a_n} = K,$$

a constant with $0 < K < \infty$, show that $\sum b_n$ converges or diverges with $\sum a_n$.
Hint. If $\sum a_n$ converges, use

$$b_n' = \frac{1}{2K} b_n.$$

If $\sum a_n$ diverges, use

$$b_n'' = \frac{2}{K} b_n.$$

5.2.3 Show that the complete d'Alembert ratio test follows directly from Kummer's test with $a_i = 1$.

5.2.4 Show that Raabe's test is indecisive for $P = 1$ by establishing that $P = 1$ for the series

(a) $u_n = \dfrac{1}{n \ln n}$ and that this series diverges.

(b) $u_n = \dfrac{1}{n(\ln n)^2}$ and that this series converges.

Note. By direct addition $\sum_2^{100,000} [n(\ln n)^2]^{-1} = 2.02288$. The remainder of the series $n > 10^5$ yields 0.08686 by the integral comparison test. The total, then, 2 to ∞, is 2.1097.

5.2.5 Gauss's test is often given in the form of a test of the ratio

$$\frac{u_n}{u_{n+1}} = \frac{n^2 + a_1 n + a_0}{n^2 + b_1 n + b_0}.$$

For what values of the parameters a_1 and b_1 is there convergence? Divergence?

ANS. Convergent for $a_1 - b_1 > 1$,
divergent for $a_1 - b_1 \le 1$.

5.2.6 Test for convergence

(a) $\displaystyle\sum_{n=2}^{\infty} (\ln n)^{-1}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n(2n-1)}$.

(d) $\displaystyle\sum_{n=1}^{\infty} [n(n+1)]^{-1/2}$.

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1}$.

5.2.7 Test for convergence

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/n}}$

5.2.8 For what values of p and q will the following series converge?

$$\sum_{n=2}^{\infty} \frac{1}{n^p (\ln n)^q},$$

ANS. Convergent for $\begin{cases} p > 1, & \text{all } q, \\ p = 1, & q > 1, \end{cases}$
divergent for $\begin{cases} p < 1, & \text{all } q, \\ p = 1, & q \le 1. \end{cases}$

5.2.9 Determine the range of convergence for Gauss's hypergeometric series

$$F(\alpha, \beta, \gamma; x) = 1 + \frac{\alpha\beta}{1!\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)} x^2 + \cdots.$$

Hint. Gauss developed Gauss's test for the specific purpose of establishing the convergence of this series.

ANS. Convergent for $-1 < x < 1$ and $x = \pm 1$ if $\gamma > \alpha + \beta$.

5.2.10 A simple machine calculation yields

$$\sum_{n=1}^{100} n^{-3} = 1.202\,007.$$

Show that

$$1.202\,056 \le \sum_{n=1}^{\infty} n^{-3} \le 1.202\,057.$$

Hint. Use integrals to set upper and lower bounds on $\sum_{n=101}^{\infty} n^{-3}$.
Comment. A more exact value for summation $\sum_1^{\infty} n^{-3}$ is $1.202\,056\,903\ldots$.

5.2.11 Set upper and lower bounds on $\sum_{n=1}^{1,000,000} n^{-1}$, assuming that (a) the Euler–Mascheroni constant is known.

ANS. $14.392\,726 < \sum_{n=1}^{1,000,000} n^{-1} < 14.392\,727$.

(b) The Euler–Mascheroni constant is unknown.

5.2.12 Given $\sum_{n=1}^{1,000} n^{-1} = 7.485\,470\ldots$, set upper and lower bounds on the Euler–Mascheroni constant.

ANS. $0.5767 < \gamma < 0.5778$.

5.2.13 (From Olbers's paradox.) Assume a static universe in which the stars are uniformly distributed. Divide all space into shells of constant thickness; the stars in any one shell by themselves subtend a solid angle of ω_0. *Allowing for the blocking out of distant stars by nearer stars*, show that the total net solid angle subtended by all stars, shells extending to infinity, is *exactly* 4π. (Therefore the night sky should be ablaze with light.)

5.2.14 Test for convergence

$$\sum_{n=1}^{\infty} \left[\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right]^2 = \frac{1}{4} + \frac{9}{64} + \frac{25}{256} + \cdots.$$

5.2.15 The Legendre series, $\sum_{j\,\text{even}} u_j(x)$, satisfies the recurrence relations

$$u_{j+2}(x) = \frac{(j+1)(j+2) - l(l+1)}{(j+2)(j+3)} x^2 u_j(x),$$

in which the index j is even and l is some constant (but, in this problem, *not* a nonnegative odd integer). Find the range of values of x for which this Legendre series is convergent. Test the end points carefully.

ANS. $-1 < x < 1$.

5.2.16 A series solution (Section 8.5) of the Chebyshev equation leads to successive terms having the ratio

$$\frac{u_{j+2}(x)}{u_j(x)} = \frac{(k+j)^2 - n^2}{(k+j+1)(k+j+2)}x^2.$$

with $k = 0$ and $k = 1$. Test for convergence at $x = \pm 1$. *ANS.* Convergent.

5.2.17 A series solution for the ultraspherical (Gegenbauer) function $C_n^\alpha(x)$ leads to the recurrence

$$a_{j+2} = a_j \frac{(k+j)(k+j+2\alpha) - n(n+2\alpha)}{(k+j+1)(k+j+2)}.$$

Investigate the convergence of each of these series at $x = \pm 1$ as a function of the parameter α.

ANS. Convergent for $\alpha < \frac{1}{2}$;
divergent for $\alpha \geq \frac{1}{2}$.

5.2.18 A series expansion of the incomplete beta function (Section 10.4) yields

$$B_x(p,q) = x^p\left\{\frac{1}{p} + \frac{1-q}{p+1}x + \frac{(1-q)(2-q)}{2!(p+2)}x^2 + \cdots + \frac{(1-q)(2-q)\cdots(n-q)}{n!(p+n)}x^n + \cdots\right\}.$$

Given that $0 \leq x \leq 1$, $p > 0$, and $q > 0$, test this series for convergence. What happens at $x = 1$?

5.2.19 Show that the following series is convergent.

$$\sum_{s=0}^{\infty} \frac{(2s-1)!!}{(2s)!!(2s+1)}$$

Note. $(2s-1)!! = (2s-1)(2s-3)\cdots 3\cdot 1$ with $(-1)!! = 1$. $(2s)!! = (2s)(2s-2)\cdots 4\cdot 2$ with $0!! = 1$. The series appears as a series expansion of $\sin^{-1}(1)$ and equals $\pi/2$.

5.2.20 Show how to combine $\zeta(2) = \sum_{n=1}^{\infty} n^{-2}$ with α_1 and α_2 to obtain a series converging as n^{-4}.
Note. $\zeta(2)$ is actually available in closed form: $\zeta(2) = \pi^2/6$ (see Section 5.9).

5.2.21 The convergence improvement of Example 5.2.5 may be carried out more expediently (in this special case) by putting α_2 into a more symmetric form: Replacing n by $n-1$, we have

$$\alpha_2' = \sum_{n=2}^{\infty} \frac{1}{(n-1)n(n+1)} = \frac{1}{4}.$$

(a) Combine $\zeta(3)$ and α_2' to obtain convergence as n^{-5}.
(b) Let α_4' be α_4 with $n \to n-2$. Combine $\zeta(3)$, α_2', and α_4' to obtain convergence as n^{-7}.
(c) If $\zeta(3)$ is to be calculated to 6 decimal accuracy (error 5×10^{-7}), how many terms are required for $\zeta(3)$ alone? combined as in part (a)? combined as in part (b)?

Note. The error may be estimated using the corresponding integral.

ANS. (a) $\zeta(3) = \frac{5}{4} - \sum_{n=2}^{\infty} \frac{1}{n^3(n^2-1)}$.

5.2.22 Catalan's constant ($\beta(2)$ of AMS-55, Chapter 23) is defined by

$$\beta(2) = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-2} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2}\cdots.$$

Calculate $\beta(2)$ to six-digit accuracy.
Hint. The rate of convergence is enhanced by pairing the terms:

$$\frac{1}{(4k-1)^2} - \frac{1}{(4k+1)^2} = \frac{16k}{(16k^2-1)^2}.$$

If you have carried enough digits in your series summation, $\sum_{k=1}^{N} 16k/(16k^2-1)^2$, additional significant figures may be obtained by setting upper and lower bounds on the tail of the series, $\sum_{k=N+1}^{\infty}$. These bounds may be set by comparison with integrals as in the Maclaurin integral test.

ANS. $\beta(2) = 0.9159\,6559\,4177\ldots$.

5.3 ALTERNATING SERIES

In Section 5.2 we limited ourselves to series of positive terms. Now, in contrast, we consider infinite series in which the signs alternate. The partial cancellation due to alternating signs makes convergence more rapid and much easier to identify. We shall prove the Leibnitz criterion, a general condition for the convergence of an alternating series.

Leibnitz Criterion

Consider the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ with $a_n > 0$. If a_n is monotonic decreasing (for sufficiently large n) and $\lim_{n\to\infty} a_n = 0$, then the series converges.

To prove this, we examine the even partial sums

$$s_{2n} = a_1 - a_2 + a_3 - \cdots - a_{2n},$$
$$s_{2n+2} = s_{2n} + (a_{2n+1} - a_{2n+2}). \tag{5.51}$$

Since $a_{2n+1} > a_{2n+2}$, we have

$$s_{2n+2} > s_{2n}. \tag{5.52}$$

On the other hand,

$$s_{2n+2} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - a_{2n+2}. \tag{5.53}$$

Hence, with each pair of terms $a_{2p} - a_{2p+1} > 0$,

$$s_{2n+2} < a_1. \tag{5.54}$$

With the even partial sums bounded $s_{2n} < s_{2n+2} < a_1$ and the terms a_n decreasing monotonically and approaching zero, this alternating series converges.

One further important result can be extracted from the partial sums. From the difference between the series limit S and the partial sum s_n

$$S - s_n = a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots$$
$$= a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - \cdots \tag{5.55}$$

or

$$S - s_n < a_{n+1}. \tag{5.56}$$

Equation 5.56 says that the error in cutting off an alternating series after n terms is less than a_{n+1}, the first term dropped. A knowledge of the error obtained this way may be of great practical importance.

Absolute Convergence

Given a series of terms u_n in which u_n may vary in sign, if $\sum |u_n|$ converges, then $\sum u_n$ is said to be absolutely convergent. If $\sum u_n$ converges but $\sum |u_n|$ diverges, the convergence is called conditional.

The alternating harmonic series is a simple example of this conditional convergence. We have

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \cdots, \tag{5.57}$$

convergent by the Leibnitz criterion; but

$$\sum_{n=1}^{\infty} n^{-1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots \tag{5.58}$$

has been shown to be divergent in Sections 5.1 and 5.2.

The reader will note that all the tests developed in Section 5.2 assume a series of positive terms. Therefore all the tests in that section guarantee absolute convergence.

EXERCISES

5.3.1 (a) From the electrostatic two hemisphere problem (Exercise 12.3.20) we obtain the series

$$\sum_{s=0}^{\infty} (-1)^s (4s+3) \frac{(2s-1)!!}{(2s+2)!!}.$$

Test for convergence.

(b) The corresponding series for the surface charge density is

$$\sum_{s=0}^{\infty} (-1)^s (4s+3) \frac{(2s-1)!!}{(2s)!!}.$$

Test for convergence. The !! notation is explained in Section 10.1.

5.3.2 Show by direct numerical computation that the sum of the first 10 terms of

$$\lim_{x \to 1} \ln(1+x) = \ln 2 = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-1}$$

differs from $\ln 2$ by less than the eleventh term: $\ln 2 = 0.69314\,71806\ldots$.

5.3.3 In Exercise 5.2.9 the hypergeometric series is shown convergent for $x = \pm 1$, if $\gamma > \alpha + \beta$. Show that there is conditional convergence for $x = -1$ for γ down to $\gamma > \alpha + \beta - 1$.

Hint. The asymptotic behavior of the factorial function is given by Stirling's series, Section 10.3.

5.4 ALGEBRA OF SERIES

The establishment of absolute convergence is important because it can be proved that absolutely convergent series may be handled according to the ordinary familiar rules of algebra or arithmetic.

1. If an infinite series is absolutely convergent, the series sum is independent of the order in which the terms are added.
2. The series may be multiplied with another absolutely convergent series. The limit of the product will be the product of the individual series limits. The product series, a double series, will also converge absolutely.

No such guarantees can be given for conditionally convergent series. Again consider the alternating harmonic series. If we write

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = 1 - (\tfrac{1}{2} - \tfrac{1}{3}) - (\tfrac{1}{4} - \tfrac{1}{5}) - \cdots, \tag{5.59}$$

it is clear that the sum

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1} < 1. \tag{5.60}$$

However, if we rearrange the terms slightly, we may make the alternating harmonic series converge to $\frac{3}{2}$. We regroup the terms of Eq. 5.59, taking

$$\begin{aligned}(1 + \tfrac{1}{3} + \tfrac{1}{5}) - (\tfrac{1}{2}) + (\tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{15}) - (\tfrac{1}{4})\\ + (\tfrac{1}{17} + \cdots + \tfrac{1}{25}) - (\tfrac{1}{6}) + (\tfrac{1}{27} + \cdots + \tfrac{1}{35}) - (\tfrac{1}{8}) + \cdots.\end{aligned} \tag{5.61}$$

Treating the terms grouped in parenthesis as single terms for convenience, we obtain the partial sums

$s_1 = 1.5333$	$s_2 = 1.0333$
$s_3 = 1.5218$	$s_4 = 1.2718$
$s_5 = 1.5143$	$s_6 = 1.3476$
$s_7 = 1.5103$	$s_8 = 1.3853$
$s_9 = 1.5078$	$s_{10} = 1.4078$

From this tabulation of s_n and the plot of s_n versus n in Fig. 5.3 the convergence to $\frac{3}{2}$ is fairly clear. We have rearranged the terms, taking positive terms until the partial sum was equal to or greater than $\frac{3}{2}$, then adding in negative terms until the partial sum just fell below $\frac{3}{2}$, and so on. As the series extends to infinity, all original terms will eventually appear, but the partial sums of this rearranged alternating harmonic series converge to $\frac{3}{2}$.

By a suitable rearrangement of terms a conditionally convergent series may be made to converge to any desired value or even to diverge. This statement is sometimes given as Riemann's theorem. Obviously, conditionally convergent series must be treated with caution.

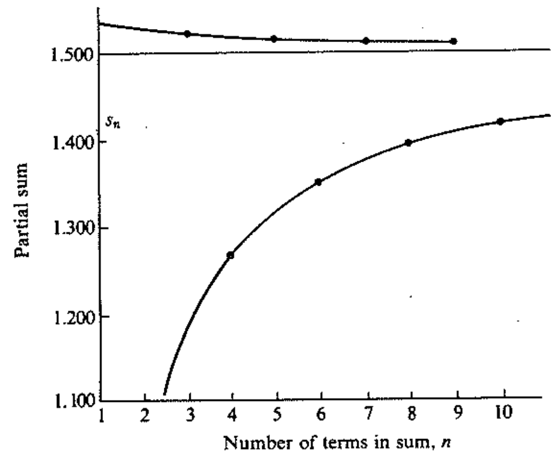


FIG. 5.3 Alternating harmonic series—terms rearranged to give convergence to 1.5

Improvement of Convergence, Rational Approximations

The series

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^n/n, \qquad -1 < x \le 1, \tag{5.61a}$$

converges very slowly as x approaches $+1$. The *rate* of convergence may be improved substantially by multiplying both sides of Eq. 5.61*a* by a polynomial and adjusting the polynomial coefficients to cancel the more slowly converging portions of the series. Consider the simplest possibility: Multiply $\ln(1+x)$ by $1 + a_1 x$.

$$(1 + a_1 x)\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^n/n + a_1 \sum_{n=1}^{\infty} (-1)^{n-1} x^{n+1}/n.$$

Combining the two series on the right term by term, we obtain

$$\begin{aligned}(1 + a_1 x)\ln(1+x) &= x + \sum_{n=2}^{\infty} (-1)^{n-1}\left(\frac{1}{n} - \frac{a_1}{n-1}\right)x^n \\ &= x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{n(1-a_1) - 1}{n(n-1)} x^n.\end{aligned}$$

Clearly, if we take $a_1 = 1$, the n in the numerator disappears and our combined series converges as n^{-2}.

Continuing this process, we find that $(1 + 2x + x^2)\ln(1+x)$ vanishes as n^{-3}, $(1 + 3x + 3x^2 + x^3)\ln(1+x)$ vanishes as n^{-4}. In effect we are shifting from a simple series expansion of Eq. 5.61*a* to a rational fraction representation in which the function $\ln(1+x)$ is represented by the ratio of a series and a polynomial:

$$\ln(1+x) = \frac{x + \sum_{n=2}^{\infty} (-1)^n x^n/[n(n-1)]}{1+x}.$$

Such rational approximations may be both compact and accurate. The SSP computer subroutines make extensive use of such approximations.

Rearrangement of Double Series

Another aspect of the rearrangement of series appears in the treatment of double series (Fig. 5.4):

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{n,m}.$$

Let us substitute

$$n = q \ge 0,$$
$$m = p - q \ge 0,$$
$$(q \le p).$$

This results in the identity

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{n,m} = \sum_{p=0}^{\infty} \sum_{q=0}^{p} a_{q,p-q} \tag{5.62}$$

The summation over p and q of Eq. 5.62 is illustrated in Fig. 5.5. The substitution

	$m = 0$	1	2	3
$n = 0$	a_{00}	a_{01}	a_{02}	a_{03}
1	a_{10}	a_{11}	a_{12}	a_{13}
2	a_{20}	a_{21}	a_{22}	a_{23}
3	a_{30}	a_{31}	a_{32}	a_{33}

FIG. 5.4 Double series—summation over n indicated by vertical dashed lines

	$p = 0$	1	2	3
$q = 0$	a_{00}	a_{01}	a_{02}	a_{03}
1		a_{10}	a_{11}	a_{12}
2			a_{20}	a_{21}
3				a_{30}

FIG. 5.5 Double series—again, the first summation is represented by vertical dashed lines but these vertical lines correspond to diagonals in Fig. 5.4.

	$r=0$	1	2	3	4
$s=0$	a_{00}	a_{01}	a_{02}	a_{03}	a_{04}
1			a_{10}	a_{11}	a_{12}
2					a_{20}

FIG. 5.6 Double series. The summation over s corresponds to a summation along the almost horizontal slanted lines in Fig. 5.4.

$$n = s \geq 0,$$
$$m = r - 2s \geq 0,$$
$$\left(s \leq \frac{r}{2}\right),$$

leads to

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{n,m} = \sum_{r=0}^{\infty}\sum_{s=0}^{[r/2]} a_{s,r-2s} \tag{5.63}$$

with $[r/2] = r/2$ for r even, $(r-1)/2$ for r odd. The summation over r and s of Eq. 5.63 is shown in Fig. 5.6. Equations 5.62 and 5.63 are clearly rearrangements of the array of coefficients a_{nm}, rearrangements that are valid as long as we have absolute convergence.

The combination of Eqs. 5.62 and 5.63,

$$\sum_{p=0}^{\infty}\sum_{q=0}^{p} a_{q,p-q} = \sum_{r=0}^{\infty}\sum_{s=0}^{[r/2]} a_{s,r-2s} \tag{5.64}$$

is used in Section 12.1 in the determination of the series form of the Legendre polynomials.

EXERCISES

5.4.1 Given the series (derived in Section 5.6)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots, \qquad -1 < x \leq 1,$$

show that

(a) $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \cdots, \qquad -1 \leq x < 1.$

(b) $\ln\left(\dfrac{1+x}{1-x}\right) = 2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots\right), \qquad -1 < x < 1.$

The original series, $\ln(1+x)$, appears in an analysis of binding energy in crystals. It is $\frac{1}{2}$ the Madelung constant ($2\ln 2$) for a chain of atoms. The second series (b) is useful in normalizing the Legendre polynomials (Section 12.3) and in developing a second solution for Legendre's differential equation (Section 12.10).

5.4.2 Determine the values of the coefficients a_1, a_2, and a_3 that will make $(1 + a_1x + a_2x^2 + a_3x^3)\ln(1+x)$ converge as n^{-4}. Find the resulting series.

5.4.3 Show that

(a) $\sum_{n=2}^{\infty} [\zeta(n) - 1] = 1.$

(b) $\sum_{n=2}^{\infty} (-1)^n[\zeta(n) - 1] = \frac{1}{2},$

where $\zeta(n)$ is the Riemann zeta function.

5.4.4 Write a program that will rearrange the terms of the alternating harmonic series to make the series converge to 1.5. Group your terms as indicated in Eq. 5.61. List the first 100 successive partial sums that just climb above 1.5 or just drop below 1.5, and list the new terms included in each such partial sum.

ANS.

n	s_n
1	1.5333
2	1.0333
3	1.5218
4	1.2718
5	1.5143

5.5 SERIES OF FUNCTIONS

We extend our concept of infinite series to include the possibility that each term u_n may be a function of some variable, $u_n = u_n(x)$. Numerous illustrations of such series of functions appear in Chapters 11 to 14. The partial sums become functions of the variable x

$$s_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x), \tag{5.65}$$

as does the series sum, defined as the limit of the partial sums

$$\sum_{n=1}^{\infty} u_n(x) = S(x) = \lim_{n\to\infty} s_n(x). \tag{5.66}$$

So far we have concerned ourselves with the behavior of the partial sums as a function of n. Now we consider how the foregoing quantities depend on x. The key concept here is that of uniform convergence.

Uniform Convergence

If for any small $\varepsilon > 0$ there exists a number N, *independent of* x in the interval $[a,b]$ $(a \leq x \leq b)$ such that

$$|S(x) - s_n(x)| < \varepsilon', \qquad \text{for all } n \geq N, \tag{5.67}$$

the series is said to be uniformly convergent in the interval $[a,b]$. This says that for our series to be uniformly convergent, it must be possible to find a

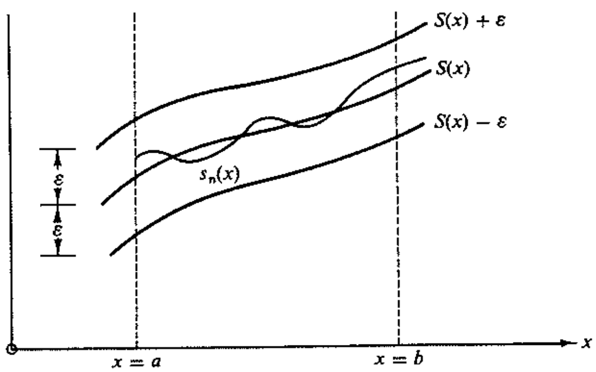


FIG. 5.7 Uniform convergence

finite N so that the tail of the infinite series, $\left|\sum_{i=N+1}^{\infty} u_i(x)\right|$, will be less than an arbitrarily small ε for all x in the given interval.

This condition, Eq. 5.67, which defines uniform convergence, is illustrated in Fig. 5.7. The point is that no matter how small ε is taken to be we can always choose n large enough so that the absolute magnitude of the difference between $S(x)$ and $s_n(x)$ is less than ε for all x, $a \le x \le b$. If this cannot be done, then $\sum u_n(x)$ is *not* uniformly convergent in $[a, b]$.

EXAMPLE 5.5.1

$$\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{x}{[(n-1)x+1][nx+1]}. \tag{5.68}$$

The partial sum $s_n(x) = nx(nx+1)^{-1}$ as may be verified by mathematical induction. By inspection this expression for $s_n(x)$ holds for $n = 1, 2$. We assume it holds for n terms and then prove it holds for $n + 1$ terms.

$$\begin{aligned} s_{n+1}(x) &= s_n(x) + \frac{x}{[nx+1][(n+1)x+1]} \\ &= \frac{nx}{[nx+1]} + \frac{x}{[nx+1][(n+1)x+1]} \\ &= \frac{(n+1)x}{(n+1)x+1}, \end{aligned}$$

completing the proof.

Letting n approach infinity, we obtain

$$S(0) = \lim_{n\to\infty} s_n(0) = 0,$$

$$S(x \neq 0) = \lim_{n\to\infty} s_n(x \neq 0) = 1.$$

We have a discontinuity in our series limit at $x = 0$. However, $s_n(x)$ is a continuous function of x, $0 \le x \le 1$, for all finite n. Equation 5.67 with ε sufficiently small, will be violated for all *finite* n. Our series does not converge uniformly.

Weierstrass M Test

The most commonly encountered test for uniform convergence is the Weierstrass M test. If we can construct a series of numbers $\sum_1^{\infty} M_i$, in which $M_i \ge |u_i(x)|$ for all x in the interval $[a, b]$ and $\sum_1^{\infty} M_i$ is convergent, our series $\sum_1^{\infty} u_i(x)$ will be *uniformly* convergent in $[a, b]$.

The proof of this Weierstrass M test is direct and simple. Since $\sum_i M_i$ converges, some number N exists such that for $n + 1 \ge N$,

$$\sum_{i=n+1}^{\infty} M_i < \varepsilon. \tag{5.69}$$

This follows from our definition of convergence. Then, with $|u_i(x)| \le M_i$ for all x in the interval $a \le x \le b$,

$$\sum_{i=n+1}^{\infty} |u_i(x)| < \varepsilon. \tag{5.70}$$

Hence

$$|S(x) - s_n(x)| = \left|\sum_{i=n+1}^{\infty} u_i(x)\right| < \varepsilon, \tag{5.71}$$

and by definition $\sum_{i=1}^{\infty} u_i(x)$ is uniformly convergent in $[a, b]$. Since we have specified absolute values in the statement of the Weierstrass M test, the series $\sum_{i=1}^{\infty} u_i(x)$ is also seen to be *absolutely* convergent.

The reader should note carefully that uniform convergence and absolute convergence are independent properties. Neither implies the other. For specific examples,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+x^2}, \qquad -\infty < x < \infty \tag{5.72}$$

and

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} = \ln(1+x), \qquad 0 \le x \le 1 \tag{5.73}$$

converge uniformly in the indicated intervals but do not converge absolutely. On the other hand,

$$\begin{aligned} \sum_{n=0}^{\infty} (1-x)x^n &= 1, \qquad 0 \le x < 1 \\ &= 0, \qquad x = 1 \end{aligned} \tag{5.74}$$

converges absolutely but does not converge uniformly in $[0, 1]$.

From the definition of uniform convergence we may show that any series

$$f(x) = \sum_{n=1}^{\infty} u_n(x) \tag{5.75}$$

cannot converge uniformly in any interval that includes a discontinuity of $f(x)$.

Since the Weierstrass M test establishes both uniform and absolute convergence, it will necessarily fail for series that are uniformly but conditionally convergent.

Abel's Test

A somewhat more delicate test for uniform convergence has been given by Abel. If

$$u_n(x) = a_n f_n(x),$$

$$\sum a_n = A, \qquad \text{convergent,}$$

and the functions $f_n(x)$ are monotonic $[f_{n+1}(x) \le f_n(x)]$ and bounded, $0 \le f_n(x) \le M$, for all x in $[a, b]$, then $\sum u_n(x)$ converges uniformly in $[a, b]$.

This test is especially useful in analyzing power series (compare Section 5.7). Details of the proof of Abel's test and other tests for uniform convergence are given in the references listed at the end of this chapter.

Uniformly convergent series have three particularly useful properties.

1. If the individual terms $u_n(x)$ are continuous, the series sum

$$f(x) = \sum_{n=1}^{\infty} u_n(x) \tag{5.76}$$

is also continuous.

2. If the individual terms $u_n(x)$ are continuous, the series may be integrated term by term. The sum of the integrals is equal to the integral of the sum.

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx. \tag{5.77}$$

3. The derivative of the series sum $f(x)$ equals the sum of the individual term derivatives,

$$\frac{d}{dx} f(x) = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x), \tag{5.78}$$

provided the following conditions are satisfied.

$$u_n(x) \quad \text{and} \quad \frac{du_n(x)}{dx} \quad \text{are continuous in } [a, b].$$

$$\sum_{n=1}^{\infty} \frac{du_n(x)}{dx} \quad \text{is uniformly convergent in } [a, b].$$

Term-by-term integration of a uniformly convergent series[1] requires only continuity of the individual terms. This condition is almost always satisfied in physical applications. Term-by-term differentiation of a series if often not valid because more restrictive conditions must be satisfied. Indeed, we shall en-

[1] Term-by-term integration *may* also be valid in the absence of uniform convergence.

counter cases in Chapter 14, Fourier Series, in which term-by-term differentiation of a uniformly convergent series leads to a divergent series.

EXERCISES

5.5.1 Find the range of *uniform* convergence of

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^x}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^x}$

ANS. (a) $1 \le x < \infty$.
(b) $1 < s \le x < \infty$.

5.5.2 For what range of x is the geometric series $\sum_{n=0}^{\infty} x^n$ uniformly convergent?

ANS. $-1 < -s \le x \le s < 1$

5.5.3 For what range of positive values of x is $\sum_{n=0}^{\infty} 1/(1 + x^n)$

(a) Convergent?
(b) Uniformly convergent?

5.5.4 If the series of the coefficients $\sum a_n$ and $\sum b_n$ are absolutely convergent, show that the Fourier series

$$\sum (a_n \cos nx + b_n \sin nx)$$

is *uniformly* convergent for $-\infty < x < \infty$.

5.6 TAYLOR'S EXPANSION

This is an expansion of a function into an infinite series or into a finite series plus a remainder term. The coefficients of the successive terms of the series involve the successive derivatives of the function. We have already used Taylor's expansion in the establishment of a physical interpretation of divergence (Section 1.7) and in other sections of Chapters 1 and 2. Now we derive the Taylor expansion.

We assume that our function $f(x)$ has a continuous nth derivative[1] in the interval $a \le x \le b$. Then, integrating this nth derivative n times,

$$\int_a^x f^{(n)}(x)\,dx = f^{(n-1)}(x)\Big|_a^x = f^{(n-1)}(x) - f^{(n-1)}(a)$$

$$\begin{aligned}\int_a^x \left(\int_a^x f^{(n)}(x)\,dx \right) dx &= \int_a^x [f^{(n-1)}(x) - f^{(n-1)}(a)]\,dx \\ &= f^{(n-2)}(x) - f^{(n-2)}(a) - (x - a) f^{(n-1)}(a).\end{aligned} \tag{5.79}$$

Continuing, we obtain

[1] Taylor's expansion may be derived under slightly less restrictive conditions, compare Jeffreys and Jeffreys, *Methods of Mathematical Physics*, Section 1.133.

$$\int\!\!\int\!\!\int_a^x f^{(n)}(x)(dx)^3 = f^{(n-3)}(x) - f^{(n-3)}(a) - (x-a)f^{(n-2)}(a) - \frac{(x-a)^2}{2}f^{(n-1)}(a). \tag{5.80}$$

Finally, on integrating for the nth time,

$$\int_a^x \cdots \int f^{(n)}(x)(dx)^n = f(x) - f(a) - (x-a)f'(a) - \frac{(x-a)^2}{2!}f''(a) - \cdots - \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a). \tag{5.81}$$

Note that this expression is exact. No terms have been dropped, no approximations made. Now, solving for $f(x)$, we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n. \tag{5.82}$$

The remainder, R_n, is given by the n-fold integral

$$R_n = \int_a^x \cdots \int f^{(n)}(x)(dx)^n. \tag{5.83}$$

This remainder, Eq. 5.83, may be put into perhaps more intelligible form by using the mean value theorem of integral calculus

$$\int_a^x g(x)\,dx = (x-a)g(\xi), \tag{5.84}$$

with $a \le \xi \le x$. By integrating n times we get the Lagrangian form[2] of the remainder:

$$R_n = \frac{(x-a)^n}{n!}f^{(n)}(\xi). \tag{5.85}$$

With Taylor's expansion in this form we are not concerned with any questions of infinite series convergence. This series is finite, and the only questions concern the magnitude of the remainder.

When the function $f(x)$ is such that

$$\lim_{n\to\infty} R_n = 0, \tag{5.86}$$

Eq. 5.82 becomes Taylor's series

[2] An alternate form derived by Cauchy is

$$R_n = \frac{(x-\zeta)^{n-1}(x-a)}{(n-1)!}f^{(n)}(\zeta),$$

with $a \le \zeta \le x$.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots = \sum_{n=0}^{\infty}\frac{(x-a)^n}{n!}f^{(n)}(a).^3 \tag{5.87}$$

Our Taylor series specifies the value of a function at one point, x, in terms of the value of the function and its derivatives at a reference point, a. It is an expansion in powers of the *change* in the variable, $\Delta x = x - a$ in this case. The notation may be varied at the user's convenience. With the substitution $x \to x + h$ and $a \to x$ we have an alternate form

$$f(x+h) = \sum_{n=0}^{\infty}\frac{h^n}{n!}f^{(n)}(x).$$

When we use the operator $D = d/dx$ the Taylor expansion becomes

$$f(x+h) = \sum_{n=0}^{\infty}\frac{h^nD^n}{n!}f(x) = e^{hD}f(x).$$

(The transition to the exponential form anticipates Eq. 5.90 that follows.) An equivalent operator form of this Taylor expansion appears in Exercise 4.11.1. A derivation of the Taylor expansion in the context of complex variable theory appears in Section 6.5.

Maclaurin Theorem

If we expand about the origin ($a = 0$), Eq. 5.87 is known as Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}f^{(n)}(0). \tag{5.88}$$

An immediate application of the Maclaurin series (or the Taylor series) is in the expansion of various transcendental functions into infinite series.

EXAMPLE 5.6.1

Let $f(x) = e^x$. Differentiating, we have

$$f^{(n)}(0) = 1 \tag{5.89}$$

for all n, $n = 1, 2, 3, \ldots$. Then, by Eq. 5.88, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}. \tag{5.90}$$

[3] Note that $0! = 1$ (compare Section 10.1).

This is the series expansion of the exponential function. Some authors use this series to define the exponential function.

Although this series is clearly convergent for all x, we should check the remainder term, R_n. By Eq. 5.85 we have

$$\begin{aligned} R_n &= \frac{x^n}{n!} f^{(n)}(\xi) \\ &= \frac{x^n}{n!} e^{\xi}, \qquad 0 \le \xi \le x. \end{aligned} \tag{5.91}$$

Therefore

$$R_n \le \frac{x^n e^x}{n!} \tag{5.92}$$

and

$$\lim_{n\to\infty} R_n = 0 \tag{5.93}$$

for all *finite* values of x, which indicates that this Maclaurin expansion of e^x is valid over the range $-\infty < x < \infty$.

EXAMPLE 5.6.2

Let $f(x) = \ln(1 + x)$. By differentiating, we obtain

$$\begin{aligned} f'(x) &= (1 + x)^{-1}, \\ f^{(n)}(x) &= (-1)^{n-1}(n-1)!(1+x)^{-n}. \end{aligned} \tag{5.94}$$

The Maclaurin expansion (Eq. 5.88) yields

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + R_n \\ &= \sum_{p=1}^{n} (-1)^{p-1}\frac{(x)^p}{p} + R_n. \end{aligned} \tag{5.95}$$

In this case our remainder is given by

$$\begin{aligned} R_n &= \frac{x^n}{n!} f^{(n)}(\xi), \qquad 0 \le \xi \le x \\ &\le \frac{x^n}{n}, \qquad 0 \le \xi \le x \le 1. \end{aligned} \tag{5.96}$$

Now the remainder approaches zero as n is increased indefinitely, provided $0 \le x \le 1$.[4] As an infinite series

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}, \tag{5.97}$$

[4] This range can easily be extended to $-1 < x \le 1$ but not to $x = -1$.

which converges for $-1 < x \le 1$. The range $-1 < x < 1$ is easily established by the d'Alembert ratio test (Section 5.2). Convergence at $x = 1$ follows by the Leibnitz criterion (Section 5.3). In particular, at $x = 1$, we have

$$\begin{aligned} \ln 2 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} n^{-1}, \end{aligned}$$

the conditionally convergent alternating harmonic series.

Binomial Theorem

A second, extremely important application of the Taylor and Maclaurin expansions is the derivation of the binomial theorem for negative and/or nonintegral powers.

Let $f(x) = (1 + x)^m$, in which m may be negative and is not limited to integral values. Direct application of Eq. 5.88 gives

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + R_n. \tag{5.99}$$

For this function the remainder is

$$R_n = \frac{x^n}{n!}(1+\xi)^{m-n} \times m(m-1)\cdots(m-n+1) \tag{5.100}$$

and ξ lies between 0 and x, $0 \le \xi \le x$. Now, for $n > m$, $(1 + \xi)^{m-n}$ is a maximum for $\xi = 0$. Therefore

$$R_n \le \frac{x^n}{n!} \times m(m-1)\cdots(m-n+1). \tag{5.101}$$

Note that the m dependent factors do not yield a zero unless m is a nonnegative integer; R_n tends to zero as $n \to \infty$ if x is restricted to the range $0 \le x < 1$.

The binomial expansion therefore is shown to be

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots. \tag{5.102}$$

In other, equivalent notation

$$\begin{aligned} (1+x)^m &= \sum_{n=0}^{\infty} \frac{m!}{n!(m-n)!}x^n \\ &= \sum_{n=0}^{\infty} \binom{m}{n} x^n. \end{aligned} \tag{5.103}$$

The quantity $\binom{m}{n}$, which equals $m!/n!(m-n)!$ is called a *binomial coefficient*. Although we have only shown that the remainder vanishes,

$$\lim_{n\to\infty} R_n = 0,$$

for $0 \le x < 1$, the series in Eq. 5.102 actually may be shown to be convergent for the extended range $-1 < x < 1$. For m an integer, $(m-n)! = \pm\infty$ if $n > m$ (Section 10.1) and the series automatically terminates at $n = m$.

EXAMPLE 5.6.3 Relativistic Energy

The total relativistic energy of a particle is

$$E = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{5.104}$$

Compare this equation with the classical kinetic energy, $\frac{1}{2}mv^2$.

By Eq. 5.102 with $x = -v^2/c^2$ and $m = -\frac{1}{2}$ we have

$$E = mc^2\left[1 - \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \frac{(-1/2)(-3/2)}{2!}\left(-\frac{v^2}{c^2}\right)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}\left(-\frac{v^2}{c^2}\right)^3 + \cdots\right]$$

or

$$E = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}mv^2\cdot\frac{v^2}{c^2} + \frac{5}{16}mv^2\cdot\left(\frac{v^2}{c^2}\right)^2 + \cdots. \tag{5.105}$$

The first term, mc^2, is identified as the rest mass energy. Then

$$E_{\text{kinetic}} = \frac{1}{2}mv^2\left[1 + \frac{3}{4}\frac{v^2}{c^2} + \frac{5}{8}\left(\frac{v^2}{c^2}\right)^2 + \cdots\right]. \tag{5.106}$$

For particle velocity $v \ll c$, the velocity of light, the expression in the brackets reduces to unity and we see that the kinetic portion of the total relativistic energy agrees with the classical result.

For polynomials we can generalize the binomial expansion to

$$(a_1 + a_2 + \cdots + a_m)^n = \sum \frac{n!}{n_1!\,n_2!\cdots n_m!}a_1^{n_1}a_2^{n_2}\cdots a_m^{n_m},$$

where the summation includes all different combinations of $n_1, n_2, \ldots, n_m$ with $\sum_{i=1}^{m} n_i = n$. Here n_i and n are all integral. This generalization finds considerable use in statistical mechanics.

Maclaurin series may sometimes appear indirectly rather than by direct use of Eq. 5.88. For instance, the most convenient way to obtain the series expansion

$$\sin^{-1} x = \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot\frac{x^{2n+1}}{(2n+1)} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots, \tag{5.106a}$$

is to make use of the relation

$$\sin^{-1} x = \int_0^x \frac{dt}{(1-t^2)^{1/2}}.$$

We expand $(1-t^2)^{-1/2}$ (binomial theorem) and then integrate term by term. This term-by-term integration is discussed in Section 5.7. The result is Eq. 5.106a.

Finally, we may take the limit as $x \to 1$. The series converges by Gauss's test, Exercise 5.2.5.

Taylor Expansion—More than One Variable

If the function f has more than one independent variable, say, $f = f(x, y)$, the Taylor expansion becomes

$$\begin{aligned} f(x,y) = f(a,b) &+ (x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y} \\ &+ \frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x\,\partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right] \\ &+ \frac{1}{3!}\left[(x-a)^3\frac{\partial^3 f}{\partial x^3} + 3(x-a)^2(y-b)\frac{\partial^3 f}{\partial x^2\,\partial y}\right. \\ &\left. + 3(x-a)(y-b)^2\frac{\partial^3 f}{\partial x\,\partial y^2} + (y-b)^3\frac{\partial^3 f}{\partial y^3}\right] + \cdots, \end{aligned} \tag{5.107}$$

with all derivatives evaluated at the point (a, b). Using $\alpha_j t = x_j - x_{j0}$, we may write the Taylor expansion for m independent variables in the symbolic form

$$f(x_j) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left(\sum_{i=1}^{m}\alpha_i\frac{\partial}{\partial x_i}\right)^n f(x_k)\Bigg|_{x_k = x_{k0}} \tag{5.108}$$

A convenient vector form is

$$\psi(\mathbf{r} + \mathbf{a}) = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathbf{a}\cdot\nabla)^n\psi(\mathbf{r}). \tag{5.109}$$

EXERCISES

5.6.1 Show that

(a) $\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$,

(b) $\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$.

In Section 6.1 e^{ix} is *defined* by a series expansion such that

$$e^{ix} = \cos x + i\sin x.$$

This is the basis for the polar representation of complex quantities. As a special case we find, with $x = \pi$,

$$e^{i\pi} = -1.$$

5.6.2 Derive a series expansion of $\cot x$ in increasing powers of x by dividing $\cos x$ by $\sin x$.

Note. The resultant series that starts with $1/x$ is actually a Laurent series (Section 6.5). Although the two series for $\sin x$ and $\cos x$ were valid for all x, the convergence of the series for $\cot x$ is limited by the zeros of the denominator, $\sin x$.

5.6.3 (a) Expand $(1+x)\ln(1+x)$ in a Maclaurin series. Find the limits on x for convergence.
(b) From the results for part (a) show that

$$\ln 2 = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)}.$$

ANS. (a) $(1+x)\ln(1+x) = x + \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n(n-1)}, \quad -1 < x \le 1.$

5.6.4 The Raabe test for $\sum (n \ln n)^{-1}$ leads to

$$\lim_{n\to\infty} n\left[\frac{(n+1)\ln(n+1)}{n \ln n} - 1\right].$$

Show that this limit is unity (which means that the Raabe test here is indeterminant).

5.6.5 Show by series expansion that

$$\frac{1}{2}\ln\frac{\eta_0+1}{\eta_0-1} = \coth^{-1}\eta_0, \qquad |\eta_0| > 1.$$

This identity may be used to obtain a second solution for Legendre's equation.

5.6.6 Show that $f(x) = x^{1/2}$ (a) has no Maclaurin expansion but (b) has a Taylor expansion about any point $x_0 \neq 0$. Find the range of convergence of the Taylor expansion about $x = x_0$.

5.6.7 Let x be an approximation for a zero of $f(x)$ and Δx, the correction. Show that by neglecting terms of order $(\Delta x)^2$

$$\Delta x = -\frac{f(x)}{f'(x)}.$$

This is Newton's formula for finding a root. Newton's method has the virtues of illustrating series expansions and elementary calculus but is very treacherous. See Appendix A1 for details and an alternative.

5.6.8 Expand a function $\Phi(x, y, z)$ by Taylor's expansion. Evaluate $\bar{\Phi}$, the average value of Φ, averaged over a small cube of side a centered on the origin and show that the Laplacian of Φ is a measure of deviation of Φ from $\Phi(0, 0, 0)$.

5.6.9 The ratio of two differentiable functions $f(x)$ and $g(x)$ takes on the indeterminate form 0/0 at $x = x_0$. Using Taylor expansions prove L'Hospital's rule

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

5.6.10 With $n > 1$, show that

(a) $\frac{1}{n} - \ln\left(\frac{n}{n-1}\right) < 0,$

(b) $\frac{1}{n} - \ln\left(\frac{n+1}{n}\right) > 0.$

Use these inequalities to show that the limit defining the Euler–Mascheroni constant is finite.

5.6.11 Expand $(1 - 2tz + t^2)^{-1/2}$ in powers of t. Assume that t is small. Collect the coefficients of t^0, t^1 and t^2.

ANS. $a_0 = P_0(z) = 1,$
$a_1 = P_1(z) = z,$
$a_2 = P_2(z) = \frac{1}{2}(3z^2 - 1),$
where $a_n = P_n(z)$, the nth Legendre polynomial.

5.6.12 Using the double factorial notation of Section 10.1, show that

$$(1+x)^{-m/2} = \sum_{n=0}^{\infty} (-1)^n \frac{(m+2n-2)!!}{2^n n!(m-2)!!} x^n,$$

for $m = 1, 2, 3, \ldots.$

5.6.13 Using binomial expansions, compare the three Doppler shift formulas:

(a) $\nu' = \nu\left(1 \mp \frac{v}{c}\right)^{-1}$ moving source;

(b) $\nu' = \nu\left(1 \pm \frac{v}{c}\right)$, moving observer;

(c) $\nu' = \nu\left(1 \pm \frac{v}{c}\right)\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$, relativistic.

Note. The relativistic formula agrees with the classical formulas if terms of order v^2/c^2 can be neglected.

5.6.14 In the theory of general relativity there are various ways of relating (defining) a velocity of recession of a galaxy to its red shift, δ. Milne's model (kinematic relativity) gives

(a) $v_1 = c\delta(1 + \frac{1}{2}\delta),$

(b) $v_2 = c\delta(1 + \frac{1}{2}\delta)(1 + \delta)^{-2}$

(c) $1 + \delta = \left[\frac{1 + v_3/c}{1 - v_3/c}\right]^{1/2}.$

1. Show that for $\delta \ll 1$ (and $v_3/c \ll 1$) all three formulas reduce to $v = c\delta$.
2. Compare the three velocities through terms of order δ^2.

Note. In special relativity (with δ replaced by z), the ratio of observed wavelength λ to emitted wave length λ_0 is given by

$$\frac{\lambda}{\lambda_0} = 1 + z = \left(\frac{c+v}{c-v}\right)^{1/2}.$$

5.6.15 The relativistic sum w of two velocities u and v is given by

$$\frac{w}{c} = \frac{u/c + v/c}{1 + uv/c^2}.$$

If

$$\frac{v}{c} = \frac{u}{c} = 1 - \alpha,$$

where $0 \le \alpha \le 1$, find w/c in powers of α through terms in α^3.

5.6.16 The displacement x of a particle of rest mass m_0, resulting from a constant force $m_0 g$ along the x-axis, is

$$x = \frac{c^2}{g}\left\{\left[1 + \left(g\frac{t}{c}\right)^2\right]^{1/2} - 1\right\},$$

including relativistic effects. Find the displacement x as a power series in time t. Compare with the classical result

$$x = \tfrac{1}{2}gt^2.$$

5.6.17 By use of Dirac's relativistic theory the fine structure formula of atomic spectroscopy is given by

$$E = mc^2\left[1 + \frac{\gamma^2}{(s + n - |k|)^2}\right]^{-1/2},$$

where

$$s = (|k|^2 - \gamma^2)^{1/2}, \quad k = \pm 1, \pm 2, \pm 3, \ldots.$$

Expand in powers of γ^2 through order γ^4. ($\gamma^2 = Ze^2/\hbar c$, with Z the atomic number.) This expansion is useful in comparing the predictions of the Dirac electron theory with those of a relativistic Schrödinger electron theory. Experimental results support the Dirac theory.

5.6.18 In a head-on proton–proton collision, the ratio of the kinetic energy in the center of mass system to the incident kinetic energy is

$$R = \frac{\sqrt{2mc^2(E_k + 2mc^2)} - 2mc^2}{E_k}.$$

Find the value of this ratio of kinetic energies for
(a) $E_k \ll mc^2$ (nonrelativistic)
(b) $E_k \gg mc^2$ (extreme-relativistic)

ANS. (a) $\frac{1}{2}$, (b) $\to$ 0. The latter answer is a sort of law of diminishing returns for high energy particle accelerators (with stationary targets).

5.6.19 With binomial expansions

$$\frac{x}{1-x} = \sum_{n=1}^{\infty} x^n, \qquad \frac{x}{x-1} = \frac{1}{1-x^{-1}} = \sum_{n=0}^{\infty} x^{-n}.$$

Adding these two series yields $\sum_{n=-\infty}^{\infty} x^n = 0$.
Hopefully, we can agree that this is nonsense but what has gone wrong?

5.6.20 (a) Planck's theory of quantized oscillators led to an average energy

$$\langle\varepsilon\rangle = \frac{\sum_{n=1}^{\infty} n\varepsilon_0 \exp(-n\varepsilon_0/kT)}{\sum_{n=0}^{\infty} \exp(-n\varepsilon_0/kT)}$$

where ε_0 was a fixed energy. Identify numerator and denominator as binomial expansions and show that the ratio is

$$\langle\varepsilon\rangle = \frac{\varepsilon_0}{\exp(\varepsilon_0/kT) - 1}.$$

(b) Show that the $\langle\varepsilon\rangle$ of part (a) reduces to kT, the classical result, for $kT \gg \varepsilon_0$.

5.6.21 (a) Expand by the binomial theorem and integrate term by term to obtain the Gregory series for $\tan^{-1} x$:

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \{1 - t^2 + t^4 - t^6 + \cdots\}\,dt$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad -1 \le x \le 1.$$

(b) By comparing series expansions, show that

$$\tan^{-1} x = \frac{i}{2}\ln\left(\frac{1-ix}{1+ix}\right).$$

Hint. Compare Exercise 5.4.1.

5.6.22 In numerical analysis it is often convenient to approximate $d^2\psi(x)/dx^2$ by

$$\frac{d^2}{dx^2}\psi(x) \approx \frac{1}{h^2}[\psi(x+h) - 2\psi(x) + \psi(x-h)].$$

Find the error in this approximation.

ANS. Error $= \dfrac{h^4}{12}\psi^{(4)}(x)$.

5.6.23 You have a function $y(x)$ tabulated at equally spaced values of the argument

$$\begin{cases} y_n = y(x_n) \\ x_n = x + nh. \end{cases}$$

Show that the linear combination

$$\frac{1}{12h}\{-y_2 + 8y_1 - 8y_{-1} + y_{-2}\}$$

yields

$$y_0' - \frac{h^4}{60}y^{(5)} + \cdots.$$

Hence this linear combination yields y_0' if $(h^4/60)y^{(5)}$ and higher powers of h and higher derivatives of $y(x)$ are negligible.

5.6.24 In a numerical integration of a partial differential equation the three-dimensional Laplacian is replaced by

$$\begin{aligned}\nabla^2\psi(x,y,z) \to h^{-2}[&\psi(x+h,y,z) + \psi(x-h,y,z)\\ &+ \psi(x,y+h,z) + \psi(x,y-h,z) + \psi(x,y,z+h)\\ &+ \psi(x,y,z-h) - 6\psi(x,y,z)].\end{aligned}$$

Determine the error in this approximation. Here h is the step size, the distance between adjacent points in the x-, y-, or z-direction.

5.6.25 Using double precision, calculate e from its Maclaurin series.
Note. This simple, direct approach is the best way of calculating e to high accuracy. Sixteen terms give e to 16 significant figures. The reciprocal factorials give very rapid convergence.

5.7 POWER SERIES

The power series is a special and extremely useful type of infinite series of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} a_n x^n, \tag{5.110}$$

where the coefficients a_i are constants, independent of x.[1]

Convergence

Equation 5.110 may readily be tested for convergence by either the Cauchy root test or the d'Alembert ratio test (Section 5.2). If

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = R^{-1}, \tag{5.111}$$

the series converges for $-R < x < R$. This is the interval or radius of convergence. Since the root and ratio tests fail when the limit is unity, the end points of the interval require special attention.

For instance, if $a_n = n^{-1}$, then $R = 1$ and, from Sections 5.1, 5.2, and 5.3, the series converges for $x = -1$ but diverges for $x = +1$. If $a_n = n!$, then $R = 0$ and the series diverges for all $x \neq 0$.

Uniform and Absolute Convergence

Suppose our power series (Eq. 5.110) has been found convergent for $-R < x < R$; then it will be uniformly and absolutely convergent in any *interior* interval, $-S \leq x \leq S$, where $0 < S < R$.

This may be proved directly by the Weierstrass M test (Section 5.5) by using $M_i = |a_i| S^i$.

Continuity

Since each of the terms $u_n(x) = a_n x^n$ is a continuous function of x and $f(x) = \sum a_n x^n$ converges uniformly for $-S \leq x \leq S$, $f(x)$ must be a continuous function in the interval of uniform convergence.

This behavior is to be contrasted with the strikingly different behavior of the Fourier series (Chapter 14), in which the Fourier series is used frequently to represent discontinuous functions such as sawtooth and square waves.

Differentiation and Integration

With $u_n(x)$ continuous and $\sum a_n x^n$ uniformly convergent, we find that the differentiated series is a power series with continuous functions and the same radius of convergence as the original series. The new factors introduced by differentiation (or integration) do not affect either the root or the ratio test. Therefore our power series may be differentiated or integrated as often as desired within the interval of uniform convergence (Exercise 5.7.13).

[1] Equation 5.110 may be rewritten with $z = x + iy$, replacing x. The following sections will then yield uniform convergence, integrability, and differentiability in a region of a complex plane in place of an interval on the x-axis.

In view of the rather severe restrictions placed on differentiation (Section 5.5), this is a remarkable and valuable result.

Uniqueness Theorem

In the preceding section, using the Maclaurin series, we expanded e^x and $\ln(1+x)$ into infinite series. In the succeeding chapters functions are frequently represented or perhaps defined by infinite series. We now establish that the power-series representation is unique.

If

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad -R_a < x < R_a$$
$$= \sum_{n=0}^{\infty} b_n x^n, \qquad -R_b < x < R_b, \tag{5.112}$$

with overlapping intervals of convergence, including the origin, then

$$a_n = b_n \tag{5.113}$$

for all n; that is, we assume two (different) power-series representations and then proceed to show that the two are actually identical.

From Eq. 5.112

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n, \qquad -R < x < R, \tag{5.114}$$

where R is the smaller of R_a, R_b. By setting $x = 0$ to eliminate all but the constant terms, we obtain

$$a_0 = b_0. \tag{5.115}$$

Now, exploiting the differentiability of our power series, we differentiate Eq. 5.113, getting

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n b_n x^{n-1}. \tag{5.116}$$

We again set $x = 0$ to isolate the new constant terms and find

$$a_1 = b_1. \tag{5.117}$$

By repeating this procees n times, we get

$$a_n = b_n, \tag{5.118}$$

which shows that the two series coincide. Therefore our power-series representation is unique.

This will be a crucial point in Section 8.5, in which we use a power series to develop solutions of differential equations. This uniqueness of power series appears frequently in theoretical physics. The establishment of perturbation theory in quantum mechanics is one example. The power-series representation of functions is often useful in evaluating indeterminate forms, particularly when l'Hospital's rule may be awkward to apply (Exercise 5.7.9).

...

EXAMPLE 5.7.1

Evaluate

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}. \tag{5.119}$$

Replacing $\cos x$ by its Maclaurin series expansion, we obtain

$$\frac{1-\cos x}{x^2}=\frac{1-(1-x^2/2!+x^4/4!-\cdots)}{x^2}$$
$$=\frac{x^2/2!-x^4/4!+\cdots}{x^2}$$
$$=\frac{1}{2!}-\frac{x^2}{4!}+\cdots.$$

Letting $x\to 0$, we have

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}. \tag{5.120}$$

The uniqueness of power series means that the coefficients a_n may be identified with the derivatives in a Maclaurin series. From

$$f(x)=\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(0)x^n$$

we have

$$a_n=\frac{1}{n!}f^{(n)}(0).$$

Reversion (Inversion) of Power Series

Suppose we are given a series

$$\begin{aligned} y-y_0&=a_1(x-x_0)+a_2(x-x_0)^2+\cdots\\ &=\sum_{n=1}^{\infty}a_n(x-x_0)^n. \end{aligned} \tag{5.121}$$

This gives $(y-y_0)$ in terms of $(x-x_0)$. However, it may be desirable to have an explicit expression for $(x-x_0)$ in terms of $(y-y_0)$. We may solve Eq. 5.121 for $x-x_0$ by reversion (or inversion) of our series. Assume that

$$x-x_0=\sum_{n=1}^{\infty}b_n(y-y_0)^n, \tag{5.122}$$

with the b_n to be determined in terms of the assumed known a_n. A brute-force approach, which is perfectly adequate for the first few coefficients, is simply to substitute Eq. 5.121 into Eq. 5.122. By equating coefficients of $(x-x_0)^n$ on both sides of Eq. 5.122, since the power series is unique, we obtain

$$\begin{aligned} b_1&=\frac{1}{a_1},\\ b_2&=-\frac{a_2}{a_1^3},\\ b_3&=\frac{1}{a_1^5}(2a_2^2-a_1a_3),\\ b_4&=\frac{1}{a_1^7}(5a_1a_2a_3-a_1^2a_4-5a_2^3), \quad \text{and so on.} \end{aligned} \tag{5.123}$$

Some of the higher coefficients are listed by Dwight.[2] A more general and much more elegant approach is developed by the use of complex variables in the first and second editions of *Mathematical Methods for Physicists*.

EXERCISES

5.7.1 The classical Langevin theory of paramagnetism leads to an expression for the magnetic polarization

$$P(x)=C\left(\frac{\cosh x}{\sinh x}-\frac{1}{x}\right).$$

Expand $P(x)$ as a power series for small x (low fields, high temperature).

5.7.2 The depolarizing factor L for an oblate ellipsoid in a uniform electric field parallel to the axis of rotation is

$$L=\frac{1}{\varepsilon_0}(1+\zeta_0^2)(1-\zeta_0\cot^{-1}\zeta_0),$$

where ζ_0 defines an oblate ellipsoid in oblate spheroidal coordinates (ξ,ζ,φ). Show that

$$\lim_{\zeta_0\to\infty}L=\frac{1}{3\varepsilon_0} \quad \text{(sphere)},$$
$$\lim_{\zeta_0\to 0}L=\frac{1}{\varepsilon_0} \quad \text{(thin sheet)}.$$

5.7.3 The corresponding depolarizing factor (Exercise 5.7.2) for a prolate ellipsoid is

$$L=\frac{1}{\varepsilon_0}(\eta_0^2-1)\left(\frac{1}{2}\eta_0\ln\frac{\eta_0+1}{\eta_0-1}-1\right).$$

Show that

[2] Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, 4th ed. New York: Macmillan (1961). (Compare Formula No. 50.)

$$\lim_{\eta_0\to\infty} L = \frac{1}{3\varepsilon_0} \quad \text{(sphere)},$$
$$\lim_{\eta_0\to 1} L = 0 \quad \text{(long needle)}.$$

5.7.4 The analysis of the diffraction pattern of a circular opening involves

$$\int_0^{2\pi} \cos(c\cos\varphi)\,d\varphi.$$

Expand the integrand in a series and integrate by using

$$\int_0^{2\pi} \cos^{2n}\varphi\,d\varphi = \frac{(2n)!}{2^{2n}(n!)^2}\cdot 2\pi,$$
$$\int_0^{2\pi} \cos^{2n+1}\varphi\,d\varphi = 0.$$

The result is 2π times the Bessel function $J_0(c)$.

5.7.5 Neutrons are created (by a nuclear reaction) inside a hollow sphere of radius R. The newly created neutrons are uniformly distributed over the spherical volume. Assuming that all directions are equally probable (isotropy), what is the average distance a neutron will travel before striking the surface of the sphere? Assume straight line motion, no collisions.

(a) Show that

$$\bar r = \tfrac{3}{2}R\int_0^1\int_0^\pi \sqrt{1-k^2\sin^2\theta}\,k^2\,dk\sin\theta\,d\theta.$$

(b) Expand the integrand as a series and integrate to obtain

$$\bar r = R\left[1 - 3\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)(2n+3)}\right].$$

(c) Show that the sum of this infinite series is $\frac{1}{12}$, giving $\bar r = \frac{3}{4}R$.
Hint. Show that $s_n = \frac{1}{12} - [4(2n+1)(2n+3)]^{-1}$ by mathematical induction. Then let $n\to\infty$.

5.7.6 Given that

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x\Big|_0^1 = \frac{\pi}{4},$$

expand the integrand into a series and integrate term by term obtaining[3]

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + (-1)^n\frac{1}{2n+1} + \cdots,$$

which is Leibnitz's formula for π. Compare the convergence (or lack of it) of the integrand series and the integrated series at $x = 1$.

Leibnitz's formula converges so slowly that it is quite useless for numerical work; π has been computed to 100,000 decimals[4] by using expressions such as

[3] The series expansion of $\tan^{-1}x$ (upper limit 1 replaced by x) was discovered by James Gregory in 1671, 3 years before Leibnitz. See Peter Beckmann's entertaining and informative book, *A History of Pi*, 2nd ed. Boulder, Col.: The Golem Press, (1971).

[4] Shanks, D., and J. W. Wrench, Jr., "Computation of π to 100,000 decimals," *Math Computation* **16**, 76 (1962).

$$\pi = 24\tan^{-1}\tfrac{1}{8} + 8\tan^{-1}\tfrac{1}{57} + 4\tan^{-1}\tfrac{1}{239},$$
$$\pi = 48\tan^{-1}\tfrac{1}{18} + 32\tan^{-1}\tfrac{1}{57} - 20\tan^{-1}\tfrac{1}{239}.$$

These expressions may be verified by the use of Exercise 5.6.2.

5.7.7 Expand the incomplete factorial function

$$\int_0^x e^{-t}t^n\,dt$$

in a series of powers of x for small values of x. What is the range of convergence of the resulting series? Why was x specified to be small?

ANS.
$$\int_0^x e^{-t}t^n\,dt = x^{n+1}\left[\frac{1}{(n+1)} - \frac{x}{(n+2)} + \frac{x^2}{2!(n+3)} - \cdots \frac{(-1)^p x^p}{p!(n+p+1)} + \cdots\right].$$

5.7.8 Derive the series expansion of the incomplete beta function

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt$$
$$= x^p\left\{\frac{1}{p} + \frac{1-q}{p+1}x + \cdots + \frac{(1-q)\cdots(n-q)}{n!(p+n)}x^n + \cdots\right\}$$

for $0 \le x \le 1$, $p > 0$ and $q > 0$ (if $x = 1$).

5.7.9 Evaluate

(a) $\displaystyle\lim_{x\to 0}\frac{\sin(\tan x) - \tan(\sin x)}{x^7}$,

(b) $\displaystyle\lim_{x\to 0} x^{-n}j_n(x)$ for $n = 3$,

where $j_n(x)$ is a spherical Bessel function (Section 11.7) defined by

$$j_n(x) = (-1)^n x^n\left(\frac{d}{x\,dx}\right)^n\left(\frac{\sin x}{x}\right).$$

ANS. (a) $-\dfrac{1}{30}$,

(b) $\dfrac{1}{1\cdot 3\cdot 5\cdots(2n+1)} \to \dfrac{1}{105}$ for $n = 3$.

5.7.10 Neutron transport theory gives the following expression for the inverse neutron diffusion length of k:

$$\frac{a-b}{k}\tanh^{-1}\left(\frac{k}{a}\right) = 1.$$

By series inversion or otherwise, determine k^2 as a series of powers of b/a. Give the first two terms of the series.

ANS. $k^2 = 3ab\left(1 - \dfrac{4}{5}\dfrac{b}{a}\right)$.

5.7.11 Develop a series expansion of $\sinh^{-1}x$ in powers of x by

(a) reversion of the series for $\sinh y$,
(b) a direct Maclaurin expansion.

5.7.12 A function $f(z)$ is represented by a *descending* power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^{-n}, \qquad R \le z < \infty.$$

Show that this series expansion is unique; that is, if $f(z) = \sum_{n=0}^{\infty} b_n z^{-n}$, $R \le z < \infty$, then $a_n = b_n$ for all n.

5.7.13 A power series given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for $-R < x < R$. Show that the differentiated series and the integrated series have the same interval of convergence. (Do not bother about the end points $x = \pm R$.)

5.7.14 Assuming that $f(x)$ may be expanded in a power series about the origin, $f(x) = \sum_{n=0}^{\infty} a_n x^n$, with some nonzero range of convergence. Use the techniques employed in proving uniqueness of series to show that your assumed series is a Maclaurin series:

$$a_n = \frac{1}{n!} f^{(n)}(0).$$

5.7.15 The Klein–Nishina formula for the scattering of photons by electrons contains a term of the form

$$f(\varepsilon) = \frac{(1+\varepsilon)}{\varepsilon^2}\left[\frac{2+2\varepsilon}{1+2\varepsilon} - \frac{\ln(1+2\varepsilon)}{\varepsilon}\right].$$

Here $\varepsilon = h\nu/mc^2$, the ratio of the photon energy to the electron rest mass energy. Find

$$\lim_{\varepsilon \to 0} f(\varepsilon).$$

ANS. $\frac{4}{3}$

5.7.16 The behavior of a neutron losing energy by colliding elastically with nuclei of mass A is described by a parameter ξ_1,

$$\xi_1 = 1 + \frac{(A-1)^2}{2A} \ln\frac{A-1}{A+1}.$$

An approximation, good for large A, is

$$\xi_2 = \frac{2}{A + \frac{2}{3}}.$$

Expand ξ_1 and ξ_2 in powers of A^{-1}. Show that ξ_2 agrees with ξ_1 through $(A^{-1})^2$. Find the difference in the coefficients of the $(A^{-1})^3$ term.

5.7.17 Show that each of these two integrals equals Catalan's constant

(a) $\displaystyle\int_0^1 \arctan t \frac{dt}{t}$,

(b) $\displaystyle -\int_0^1 \ln x \frac{dx}{1+x^2}$

5.7.18 Calculate π (double precision) by each of the following arc tangent expressions:

$$\pi = 16\tan^{-1}(1/5) - 4\tan^{-1}(1/239)$$
$$\pi = 24\tan^{-1}(1/8) + 8\tan^{-1}(1/57) + 4\tan^{-1}(1/239)$$
$$\pi = 48\tan^{-1}(1/18) + 32\tan^{-1}(1/57) - 20\tan^{-1}(1/239).$$

You should obtain 16 significant figures.
Note. These formulas have been used in some of the more accurate calculations of π.[5]

5.7.19 An analysis of the Gibbs phenomenon of Section 14.5 leads to the expression

$$\frac{2}{\pi}\int_0^{\pi} \frac{\sin\xi}{\xi}\, d\xi.$$

(a) Expand the integrand in a series and integrate term by term. Find the numerical value of this expression to four significant figures.
(b) Evaluate this expression by the Gaussian quadrature (Appendix A2).

ANS. 1.178980.

5.8 ELLIPTIC INTEGRALS

Elliptic integrals are included here partly as an illustration of the use of power series and partly for their own intrinsic interest. This interest includes the occurrence of elliptic integrals in physical problems (Example 5.8.1 and Exercise 5.8.4) and applications in mathematical problems.

EXAMPLE 5.8.1 Period of a Simple Pendulum

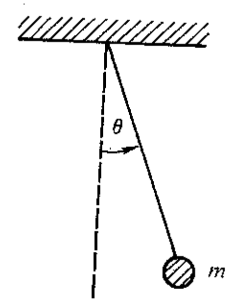
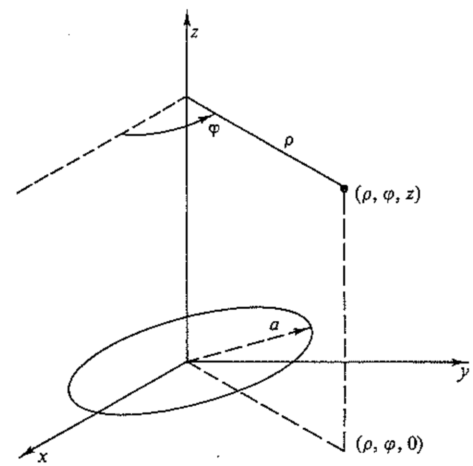


FIG. 5.8 Simple pendulum

For small amplitude oscillations our pendulum (Figure 5.8) has simple harmonic motion with a period $T = 2\pi(l/g)^{1/2}$. For a maximum amplitude θ_M large

[5] Shanks, D., and J. W. Wrench, "Computation of π to 100,000 decimals," *Math. Computation* **16**, 76 (1962).



5.8.5 An analysis of the magnetic vector potential of a circular current loop leads to the expression

$$f(k^2) = k^{-2}[(2 - k^2)K(k^2) - 2E(k^2)],$$

where $K(k^2)$ and $E(k^2)$ are the complete elliptic integrals of the first and second kinds. Show that for $k^2 \ll 1$ ($r \gg$ radius of loop)

$$f(k^2) \approx \frac{\pi k^2}{16}.$$

5.8.6 Show that

(a) $\dfrac{dE(k^2)}{dk} = \dfrac{1}{k}(E - K)$,

(b) $\dfrac{dK(k^2)}{dk} = \dfrac{E}{k(1 - k^2)} - \dfrac{K}{k}$.

Hint. For part (b) show that

$$E(k^2) = (1 - k^2)\int_0^{\pi/2} (1 - k\sin^2\theta)^{-3/2}\, d\theta$$

by comparing series expansions.

5.8.7 (a) Write a function subroutine that will compute $E(m)$ from the series expansion, Eq. 5.137.
(b) Test your function subroutine by using it to calculate $E(m)$ over the range $m = 0.0(0.1)0.9$ and comparing the result with the values given by AMS-55.

5.8.8 Repeat Exercise 5.8.7 for $K(m)$. To be written out as in Exercise 5.8.7.
Note. These series for $E(m)$, Eq. 5.137, and $K(m)$, Eq. 5.136, converge only very slowly for m near 1. More rapidly converging series for $E(m)$ and $K(m)$ exist. See Dwight's Tables of Integrals:[2] No. 773.2 and 774.2. Your computer subroutine for computing E and K probably uses polynomial approximations: AMS-55, Chapter 17.

[2] Dwight, H. B., *Tables of Integrals and Other Mathematical Data.* New York: Macmillan Co. (1947).

5.8.9 A simple pendulum is swinging with a maximum amplitude of θ_M. In the limit as $\theta_M \to 0$, the period is 1 sec. Using the elliptic integral, $K(k^2)$, $k = \sin(\theta_M/2)$ calculate the period T for $\theta_M = 0\ (10°)\ 90°$.
Caution. Some elliptic integral subroutines require $k = m^{1/2}$ as an input parameter, not m itself.

Check values.

θ	T (sec)
10°	1.00193
50°	1.05033
90°	1.18258

5.8.10 Calculate the magnetic vector potential $\mathbf{A}(\rho, \varphi, z) = \boldsymbol{\varphi}_0 A_\varphi(\rho, \varphi, z)$ of a circular current loop (Exercise 5.8.4) for the ranges $\rho/a = 2, 3, 4$, and $z/a = 0, 1, 2, 3, 4$.
Note. This elliptic integral calculation of the magnetic vector potential may be checked by an associated Legendre function calculation, Example 12.5.1.
Check value. For $\rho/a = 3$ and $z/a = 0$; $A_\varphi = 0.029023\mu_0 I$.

5.9 BERNOULLI NUMBERS, EULER–MACLAURIN FORMULA

The Bernoulli numbers were introduced by Jacques (James, Jacob) Bernoulli. There are several equivalent definitions, but extreme care must be taken, for some authors introduce variations in numbering or in algebraic signs. One relatively simple approach is to define the Bernoulli numbers by the series[1]

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}. \tag{5.144}$$

By differentiating this power series repeatedly and then setting $x = 0$, we obtain

$$B_n = \left[\frac{d^n}{dx^n}\left(\frac{x}{e^x - 1}\right)\right]_{x=0}. \tag{5.145}$$

Specifically,

$$\begin{aligned} B_1 = \frac{d}{dx}\left(\frac{x}{e^x - 1}\right)\bigg|_{x=0} &= \frac{1}{e^x - 1} - \frac{xe^x}{(e^x - 1)^2}\bigg|_{x=0} \\ &= -\frac{1}{2}, \end{aligned} \tag{5.146}$$

as may be seen by series expansion of the denominators.

Since these derivatives are awkward to evaluate, we may introduce instead a series expansion into the defining expression (Eq. 5.144) to obtain

$$\frac{1}{x}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(B_0 + B_1 x + B_2\frac{x^2}{2!} + \cdots\right) = 1. \tag{5.147}$$

Using the power-series uniqueness theorem (Section 5.7) with the coefficient of

[1] The function $x/(e^x - 1)$ may be considered a *generating function* since it generates the Bernoulli numbers. Generating functions that generate the special functions of mathematical physics appear in Chapters 11, 12, and 13.

TABLE 5.1 Bernoulli Numbers

n	B_n	B_n
0	1	1.0000 00000
1	$-\frac{1}{2}$	−0.5000 00000
2	$\frac{1}{6}$	0.1666 66667
4	$-\frac{1}{30}$	−0.0333 33333
6	$\frac{1}{42}$	0.0238 09524
8	$-\frac{1}{30}$	−0.0333 33333
10	$\frac{5}{66}$	0.0757 57576

x^0 equal to unity and the coefficient of $x^n (n \neq 0)$ equal to zero, we obtain

$$B_0 = 1$$

$$\frac{1}{2!}B_0 + B_1 = 0, \qquad B_1 = -\frac{1}{2} \tag{5.148}$$

$$\frac{1}{3!}B_0 + \frac{1}{2!}B_1 + \frac{B_2}{2!} = 0, \qquad B_2 = \frac{1}{6}. \tag{5.149}$$

Continuing, we have Table 5.1.
Further values are given in National Bureau of Standards, *Handbook of Mathematical Functions* (AMS-55).

$$B_{2n+1} = 0 \qquad n = 1, 2, 3, \ldots,$$

If the variable x in Eq. 5.144 is replaced by $2ix$ (and B_1 set equal to $-\frac{1}{2}$), we obtain an alternate (and equivalent) definition of B_{2n} by the expression

$$x \cot x = \sum_{n=0}^{\infty} (-1)^n B_{2n} \frac{(2x)^{2n}}{(2n)!}, \qquad -\pi < x < \pi. \tag{5.150}$$

Using the method of residues (Section 7.2) or working from the infinite product representation of $\sin x$ (Section 5.10), we find that

$$B_{2n} = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{p=1}^{\infty} p^{-2n}, \qquad n = 1, 2, 3, \ldots. \tag{5.151}$$

This representation of the Bernoulli numbers was discovered by Euler. It is readily seen from Eq. 5.151 that $|B_{2n}|$ increases without limit as $n \to \infty$. Numerical values have been calculated by Glaisher.[2] Illustrating the divergent behavior of the Bernoulli numbers, we have

$$\begin{aligned} B_{20} &= -5.291 \times 10^2 \\ B_{200} &= -3.647 \times 10^{215}. \end{aligned} \tag{5.152}$$

[2] Glaisher, J. W. L., "Table of the first 250 Bernoulli's numbers (to nine figures) and their logarithms (to ten figures). *Trans. Cambridge Phil. Soc.* **XII**, 390 (1871–1879).

Some authors prefer to define the Bernoulli numbers with a modified version of Eq. 5.151 by using

$$B_n = \frac{2(2n)!}{(2\pi)^{2n}} \sum_{p=1}^{\infty} p^{-2n}, \tag{5.153}$$

the subscript being just half of our subscript and all signs are positive. Again, when using other texts or references the reader must check carefully to see exactly how the Bernoulli numbers are defined.

The Bernoulli numbers occur frequently in number theory. The von Standt–Clausen theorem states that

$$B_{2n} = A_n - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3} - \cdots - \frac{1}{p_k}, \tag{5.154}$$

in which A_n is an integer and $p_1, p_2, \ldots, p_k$ are prime numbers that exceed by 1, a divisor of $2n$. It may readily be verified that this holds for

$$\begin{aligned} &B_6 (A_3 = 1, \quad p = 2, 3, 7), \\ &B_8 (A_4 = 1, \quad p = 2, 3, 5), \\ &B_{10} (A_5 = 1, \quad p = 2, 3, 11), \end{aligned} \tag{5.155}$$

and other special cases.

The Bernoulli numbers appear in the summation of integral powers of the integers,

$$\sum_{j=1}^{N} j^p, \qquad p \text{ integral},$$

and in numerous series expansion of the transcendental functions, including

$$\tan x,$$
$$\cot x,$$
$$\csc x,$$
$$\ln|\sin x|,$$
$$\ln|\cos x|,$$
$$\ln|\tan x|,$$
$$\tanh x,$$
$$\coth x,$$

and

$$\operatorname{csch} x.$$

For example,

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots + \frac{(-1)^{n-1} 2^{2n}(2^{2n} - 1)B_{2n}}{(2n)!} x^{2n-1} + \cdots. \tag{5.156}$$

TABLE 5.2 Bernoulli Functions

$B_0 = 1$
$B_1 = x - \frac{1}{2}$
$B_2 = x^2 - x + \frac{1}{6}$
$B_3 = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$
$B_4 = x^4 - 2x^3 + x^2 - \frac{1}{30}$
$B_5 = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$
$B_6 = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$
$B_n(0) = B_n$, Bernoulli number

The Bernoulli numbers are likely to come in such series expansions because of the defining equations (5.144) and (5.150) and because of their relation to the Riemann zeta function

$$\zeta(2n) = \sum_{p=1}^{\infty} p^{-2n}. \tag{5.157}$$

Bernoulli Functions

If Eq. 5.144 is generalized slightly, we have

$$\frac{xe^{xs}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(s)\frac{x^n}{n!} \tag{5.158}$$

defining the *Bernoulli functions*, $B_n(s)$. The first seven Bernoulli functions are given in Table 5.2.

From the generating function, Eq. 5.158,

$$B_n(0) = B_n, \qquad n = 0, 1, 2, \ldots, \tag{5.159}$$

the Bernoulli function evaluated at zero equals the corresponding Bernoulli number. Two particularly important properties of the Bernoulli functions follow from the defining relation: a differentiation relation

$$B_n'(s) = nB_{n-1}(s), \qquad n = 1, 2, 3, \ldots, \tag{5.160}$$

and a symmetry relation

$$B_n(1) = (-1)^n B_n(0), \qquad n = 0, 1, 2, \ldots. \tag{5.161}$$

These relations are used in the development of the Euler–Maclaurin integration formula.

Euler–Maclaurin Integration Formula

One use of the Bernoulli functions is in the derivation of the Euler–Maclaurin integration formula. This formula is used in Section 10.3 for the development of an asymptotic expression for the factorial function—Stirling's series.

The technique is repeated integration by parts using Eq. 5.160 to create new derivatives. We start with

$$\int_0^1 f(x)\,dx = \int_0^1 f(x)B_0(x)\,dx. \tag{5.162}$$

From Eq. 5.160 and Exercise 5.9.2

$$B_1'(x) = B_0(x) = 1. \tag{5.163}$$

Substituting $B_1'(x)$ into Eq. 5.162 and integrating by parts, we obtain

$$\begin{aligned}\int_0^1 f(x)\,dx &= f(1)B_1(1) - f(0)B_1(0) - \int_0^1 f'(x)B_1(x)\,dx \\ &= \frac{1}{2}[f(1) + f(0)] - \int_0^1 f'(x)B_1(x)\,dx.\end{aligned} \tag{5.164}$$

Again, using Eq. 5.160, we have

$$B_1(x) = \tfrac{1}{2}B_2'(x), \tag{5.165}$$

and integrating by parts

$$\begin{aligned}\int_0^1 f(x)\,dx = \frac{1}{2}[f(1) + f(0)] &- \frac{1}{2!}[f'(1)B_2(1) - f'(0)B_2(0)] \\ &+ \frac{1}{2!}\int_0^1 f^{(2)}(x)B_2(x)\,dx.\end{aligned} \tag{5.166}$$

Using the relations, we get

$$\begin{aligned}B_{2n}(1) &= B_{2n}(0) = B_{2n}, \qquad n = 0, 1, 2, \ldots \\ B_{2n+1}(1) &= B_{2n+1}(0) = 0, \qquad n = 1, 2, 3, \ldots\end{aligned} \tag{5.167}$$

and continuing this process, we have

$$\begin{aligned}\int_0^1 f(x)\,dx = \frac{1}{2}[f(1) + f(0)] &- \sum_{p=1}^{q} \frac{1}{(2p)!}B_{2p}[f^{(2p-1)}(1) - f^{(2p-1)}(0)] \\ &+ \frac{1}{(2q)!}\int_0^1 f^{(2q)}(x)B_{2q}(x)\,dx.\end{aligned} \tag{5.168a}$$

This is the Euler–Maclaurin integration formula. It assumes that the function $f(x)$ has the required derivatives.

The range of integration in Eq. 5.168a may be shifted from $[0, 1]$ to $[1, 2]$ by replacing $f(x)$ by $f(x + 1)$. Adding such results up to $[n - 1, n]$,

$$\begin{aligned}\int_0^n f(x)\,dx = \tfrac{1}{2}f(0) + f(1) + f(2) + \cdots + f(n-1) + \tfrac{1}{2}f(n) \\ - \sum_{p=1}^{q} \frac{1}{(2p)!}B_{2p}[f^{(2p-1)}(n) - f^{(2p-1)}(0)] \\ + \text{remainder term.}\end{aligned} \tag{5.168b}$$

The terms $\frac{1}{2}f(0) + f(1) + \cdots + \frac{1}{2}f(n)$ appear exactly as in trapezoidal integration or quadrature. The summation over p may be interpreted as a correction to the trapezoidal approximation. Equation 5.168b is the form used in Exercise 5.9.5 for summing positive powers of integers and in Section 10.3 for the derivation of Stirling's formula.

TABLE 5.3 Riemann Zeta Function

s	$\zeta(s)$
2	1.64493 40668
3	1.20205 69032
4	1.08232 32337
5	1.03692 77551
6	1.01734 30620
7	1.00834 92774
8	1.00407 73562
9	1.00200 83928
10	1.00099 45751

The Euler–Maclaurin formula is often useful in summing series by converting them to integrals.[3]

Riemann Zeta Function

This series $\sum_{p=1}^{\infty} p^{-2n}$ was used as a comparison series for testing convergence (Section 5.2) and in Eq. 5.151 as one definition of the Bernoulli numbers, B_{2n}. It also serves to define the Riemann zeta function by

$$\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s}, \qquad s > 1. \tag{5.169}$$

Table 5.3 lists the values of $\zeta(s)$ for integral s, $s = 2, 3, \ldots, 10$. Closed forms for even s appear in Exercise 5.9.6. Figure 5.10 is a plot of $\zeta(s) - 1$. An integral expression for this Riemann zeta function appears in Section 10.2 as part of the development of the gamma function.

Another interesting expression for the Riemann zeta function may be derived as follows:

$$\zeta(s)(1 - 2^{-s}) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots\right), \tag{5.170}$$

eliminating all the n^{-s}, where n is a multiple of 2. Then

$$\begin{aligned} \zeta(s)(1 - 2^{-s})(1 - 3^{-s}) = 1 &+ \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \cdots \\ &- \left(\frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \cdots\right), \end{aligned} \tag{5.171}$$

eliminating all the remaining terms in which n is a multiple of 3. Continuing, we have $\zeta(s)(1 - 2^{-s})(1 - 3^{-s})(1 - 5^{-s}) \cdots (1 - P^{-s})$, where P is a prime number, and all terms n^{-s}, in which n is a multiple of any integer up through P, are canceled out. As $P \to \infty$,

[3] Compare Boas, R. P., and C. Stutz, "Estimating Sums with Integrals," *Am. J. Phys.* 39, 745 (1971) for a number of examples.

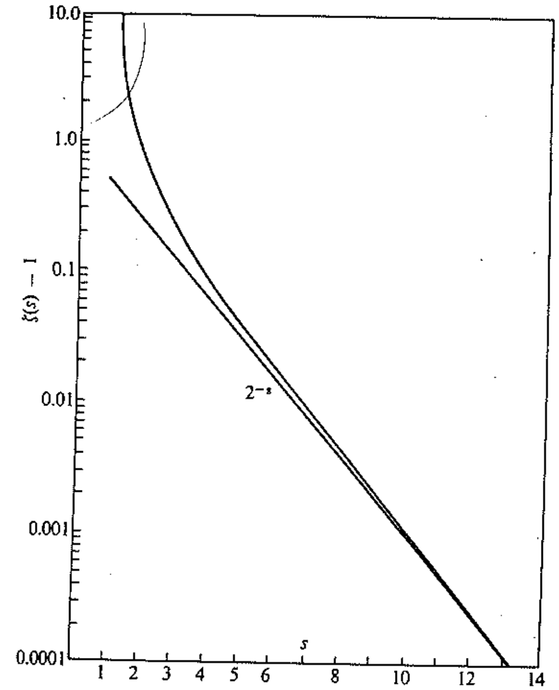


FIG. 5.10 Riemann zeta function, $\zeta(s) - 1$ versus s

$$\zeta(s)(1 - 2^{-s})(1 - 3^{-s}) \cdots (1 - P^{-s}) = \zeta(s) \prod_{P(\text{prime})=2}^{\infty} (1 - P^{-s}) = 1. \tag{5.172}$$

Therefore

$$\zeta(s) = \left[\prod_{P(\text{prime})=2}^{\infty} (1 - P^{-s})\right]^{-1}, \tag{5.173}$$

giving $\zeta(s)$ as an infinite product.[4]

This cancellation procedure has a clear application in numerical computation. Equation 5.170 will give $\zeta(s)(1 - 2^{-s})$ to the same accuracy as Eq. 5.169 gives $\zeta(s)$, but with only half as many terms. (In either case, a correction would be made for the neglected tail of the series by the Maclaurin integral test technique—replacing the series by an integral, Section 5.2.)

Along with the Riemann zeta function, AMS-55 (Chapter 23) defines three other functions of sums of reciprocal powers:

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = (1 - 2^{1-s})\zeta(s), \qquad n = 1, 2, \ldots$$

[4] This is the starting point for the extensive applications of the Riemann zeta function to the number theory. See Edwards, H. M., *Riemann's Zeta Function*. New York: Academic Press (1974).

$$\lambda(s) = \sum_{n=0}^{\infty} (2n+1)^{-s} = (1-2^{-s})\zeta(s), \qquad n = 2, 3, \ldots$$

and

$$\beta(s) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s}, \qquad n = 1, 2, \ldots.$$

From the Bernoulli numbers (Exercise 5.9.6) or Fourier series (Examples 14.3.3 and 14.3.13) special values are

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

$$\zeta(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$$

$$\eta(2) = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

$$\eta(4) = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \cdots = \frac{7\pi^4}{720}$$

$$\lambda(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

$$\lambda(4) = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

$$\beta(1) = 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$$

$$\beta(3) = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = \frac{\pi^3}{32}.$$

Catalan's constant,

$$\beta(2) = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots = 0.9159\,6559\ldots,$$

is the topic of Exercise 5.2.22.

Improvement of Convergence

If we are required to sum a convergent series $\sum_{n=1}^{\infty} a_n$ whose terms are rational functions of n, the convergence may be improved dramatically by introducing the Riemann zeta function.

EXAMPLE 5.9.1 Improvement of convergence

The problem is to evaluate the series $\sum_{n=1}^{\infty} 1/(1+n^2)$. Expanding $(1+n^2)^{-1} = n^{-2}(1+n^{-2})^{-1}$ by direct division, we have

$$(1+n^2)^{-1} = n^{-2}\left(1 - n^{-2} + n^{-4} - \frac{n^{-6}}{1+n^{-2}}\right)$$
$$= \frac{1}{n^2} - \frac{1}{n^4} + \frac{1}{n^6} - \frac{1}{n^8+n^6}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \zeta(2) - \zeta(4) + \zeta(6) - \sum_{n=1}^{\infty} \frac{1}{n^8+n^6}.$$

The ζ functions are tabulated and the remainder series converges as n^{-8}. Clearly, the process can be continued as desired. You make a choice between how much algebra you will do and how much arithmetic the computing machine will do.

Other methods for improving computational effectiveness are given at the end of Sections 5.2 and 5.4.

EXERCISES

5.9.1 Show that

$$\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} x^{2n-1}, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Hint. $\tan x = \cot x - 2\cot 2x$.

5.9.2 The Bernoulli numbers generated in Eq. 5.144 may be generalized to Bernoulli polynomials,

$$\frac{xe^{xs}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(s)\frac{x^n}{n!}.$$

Show that

$$B_0(s) = 1$$
$$B_1(s) = s - \tfrac{1}{2}$$
$$B_2(s) = s^2 - s + \tfrac{1}{6}.$$

Note that $B_n(0) = B_n$, the Bernoulli number.

5.9.3 Show that $B_n'(s) = nB_{n-1}(s), \quad n = 1, 2, 3, \ldots.$
Hint. Differentiate the equation in Exercise 5.9.2.

5.9.4 Show that

$$B_n(1) = (-1)^n B_n(0).$$

Hint. Go back to the generating function, Eq. 5.158 or Exercise 5.9.2.

5.9.5 The Euler–Maclaurin integration formula may be used for the evaluation of finite series:

$$\sum_{m=1}^{n} f(m) = \int_1^n f(x)\,dx + \frac{1}{2}f(1) + \frac{1}{2}f(n) + \frac{B_2}{2!}[f'(n) - f'(1)] + \cdots.$$

Show that

(a) $\sum_{m=1}^{n} m = \frac{1}{2}n(n+1)$.

(b) $\sum_{m=1}^{n} m^2 = \frac{1}{6}n(n+1)(2n+1)$.

(c) $\sum_{m=1}^{n} m^3 = \frac{1}{4}n^2(n+1)^2$.

(d) $\sum_{m=1}^{n} m^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$.

5.9.6 From

$$B_{2n} = (-1)^{n-1}\frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n).$$

Show that

(a) $\zeta(2) = \dfrac{\pi^2}{6}$ (d) $\zeta(8) = \dfrac{\pi^8}{9450}$

(b) $\zeta(4) = \dfrac{\pi^4}{90}$ (e) $\zeta(10) = \dfrac{\pi^{10}}{93{,}555}$

(c) $\zeta(6) = \dfrac{\pi^6}{945}$

5.9.7 Planck's black-body radiation law involves the integral

$$\int_0^\infty \frac{x^3\,dx}{e^x - 1}.$$

Show that this equals 6 $\zeta(4)$. From Exercise 5.9.6

$$\zeta(4) = \frac{\pi^4}{90}.$$

Hint. Make use of the gamma function, Chapter 10.

5.9.8 Prove that

$$\int_0^\infty \frac{x^n e^x\,dx}{(e^x - 1)^2} = n!\zeta(n).$$

Assuming n to be real, show that each side of the equation diverges if $n = 1$. Hence the preceding equation carries the condition $n > 1$. Integrals such as this appear in the quantum theory of transport effects—thermal and electrical conductivity.

5.9.9 The Bloch–Gruneissen approximation for the resistance in a monovalent metal is

$$\rho = C\frac{T^5}{\Theta^6}\int_0^{\Theta/T}\frac{x^5\,dx}{(e^x - 1)(1 - e^{-x})},$$

where Θ is the Debye temperature characteristic of the metal.

(a) For $T \to \infty$ show that

$$\rho \approx \frac{C}{4}\cdot\frac{T}{\Theta^2}.$$

(b) For $T \to 0$, show that

$$\rho \approx 5!\zeta(5)C\frac{T^5}{\Theta^6}.$$

5.9.10 Show that

(a) $\displaystyle\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{1}{2}\zeta(2)$,

(b) $\displaystyle\lim_{a\to 1}\int_0^a \frac{\ln(1-x)}{x}\,dx = \zeta(2)$.

From Exercise 5.9.6, $\zeta(2) = \pi^2/6$. Note that the integrand in part (b) diverges for $a = 1$ but that the *integrated* series is convergent.

5.9.11 The integral

$$\int_0^1 [\ln(1-x)]^2\frac{dx}{x}$$

appears in the fourth-order correction to the magnetic moment of the electron. Show that it equals 2 $\zeta(3)$.
Hint. Let $1 - x = e^{-t}$.

5.9.12 Show that

$$\int_0^\infty \frac{(\ln z)^2}{1+z^2}\,dz = 4\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\right).$$

By contour integration (Exercise 7.2.17), this may be shown equal to $\pi^3/8$.

5.9.13 For "small" values of x

$$\ln(x!) = -\gamma x + \sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}x^n,$$

where γ is the Euler–Mascheroni constant and $\zeta(n)$ the Riemann zeta function. For what values of x does this series converge?

ANS. $-1 < x \le 1$.

Note that if $x = 1$, we obtain

$$\gamma = \sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n},$$

a series for the Euler–Mascheroni constant. The convergence of this series is exceedingly slow. For actual computation of γ, other, indirect approaches are far superior (see Exercises 5.9.17, 5.10.11, and 10.5.16).

5.9.14 Show that the series expansion of $\ln(x!)$ (Exercise 5.9.13) may be written as

(a) $\displaystyle\ln(x!) = \frac{1}{2}\ln\left(\frac{\pi x}{\sin \pi x}\right) - \gamma x - \sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{2n+1}x^{2n+1}$,

(b) $\displaystyle\ln(x!) = \frac{1}{2}\ln\left(\frac{\pi x}{\sin \pi x}\right) - \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + (1-\gamma)x - \sum_{n=1}^{\infty}[\zeta(2n+1) - 1]\frac{x^{2n+1}}{2n+1}$.

Determine the range of convergence of each of these expressions.

5.9.15 Show that Catalan's constant, $\beta(2)$, may be written as

$$\beta(2) = 2\sum_{k=1}^{\infty}(4k-3)^{-2} - \frac{\pi^2}{8}.$$

Hint. $\pi^2 = 6\zeta(2)$.

5.9.16 Derive the following expansions of the Debye functions

(a) $$\int_0^x \frac{t^n\,dt}{e^t-1} = x^n\left[\frac{1}{n} - \frac{x}{2(n+1)} + \sum_{k=1}^{\infty}\frac{B_{2k}x^{2k}}{(2k+n)(2k)!}\right], \qquad |x| < 2\pi,\ n \ge 1,$$

(b) $$\int_x^{\infty} \frac{t^n\,dt}{e^t-1} = \sum_{k=1}^{\infty} e^{-kx}\left[\frac{x^n}{k} + \frac{nx^{n-1}}{k^2} + \frac{n(n-1)x^{n-2}}{k^3} + \cdots + \frac{n!}{k^{n+1}}\right],$$

$x > 0$, $n \ge 1$.

The complete integral $(0, \infty)$ equals $n!\,\zeta(n+1)$, Exercise 10.2.15.

5.9.17 Derive the following Bernoulli number series for the Euler–Mascheroni constant.

$$\gamma = \sum_{s=1}^{n} s^{-1} - \ln n - \frac{1}{2n} + \sum_{k=1} \frac{B_{2k}}{(2k)n^{2k}}.$$

Hint. Apply the Euler–Maclaurin integration formula to $f(x) = x^{-1}$ over the range $[n, N]$.

5.9.18 (a) Show that the equation $\ln 2 = \sum_{s=1}^{\infty}(-1)^{s+1}s^{-1}$. (Exercise 5.4.1) may be rewritten as

$$\ln 2 = \sum_{s=2}^{\infty} 2^{-s}\zeta(s) + \sum_{p=1}^{\infty}(2p)^{-n-1}\left[1 - \frac{1}{2p}\right]^{-1}.$$

Hint. Take the terms in pairs.

(b) Calculate ln 2 to six significant figures.

5.9.19 (a) Show that the equation $\pi/4 = \sum_{n=1}^{\infty}(-1)^{n+1}(2n-1)^{-1}$ (Exercise 5.7.6) may be rewritten as

$$\frac{\pi}{4} = 1 - 2\sum_{s=1}^{n} 4^{-2s}\zeta(2s)$$
$$- 2\sum_{p=1}^{\infty}(4p)^{-2n-2}\left[1 - \frac{1}{(4p)^2}\right]^{-1}.$$

(b) Calculate $\pi/4$ to six significant figures.

5.9.20 Write a function subprogram ZETA(N) that will calculate the Riemann zeta function for integer argument. Tabulate $\zeta(s)$ for $s = 2, 3, 4, \ldots, 20$. Check your values against Table 5.3 and AMS-55, Chapter 23.
Hint. If you simply supply the function subprogram with the values of $\zeta(2)$, $\zeta(3)$, and $\zeta(4)$, you avoid the more slowly converging series. Calculation time may be further shortened by using Eq. 5.170.

5.9.21 Calculate the logarithm (base 10) of $|B_{2n}|$, $n = 10, 20, \ldots, 100$.
Hint. Program the zeta function as a function subprogram, Exercise 5.9.20.
Check values. $\log|B_{100}| = 78.45$
$\log|B_{200}| = 215.56$

5.10 ASYMPTOTIC OR SEMICONVERGENT SERIES

Asymptotic series frequently occur in physics. In numerical computations they are employed for the accurate computation of a variety of functions. We consider here two types of integrals that lead to asymptotic series: first, an integral of the form

$$I_1(x) = \int_x^{\infty} e^{-u}f(u)\,du,$$

where the variable x appears as the lower limit of an integral. Second, we consider the form

$$I_2(x) = \int_0^{\infty} e^{-u}f\left(\frac{u}{x}\right)du,$$

with the function f to be expanded as a Taylor series (binomial series). Asymptotic series often occur as solutions of differential equations. An example of this appears in Section 11.6 as a solution of Bessel's equation.

Incomplete Gamma Function

The nature of an asymptotic series is perhaps best illustrated by a specific example. Suppose that we have the exponential integral function[1]

$$Ei(x) = \int_{-\infty}^{x} \frac{e^u}{u}\,du, \tag{5.174}$$

or

$$-Ei(-x) = \int_x^{\infty}\frac{e^{-u}}{u}\,du = E_1(x), \tag{5.175}$$

to be evaluated for large values of x. Better still, let us take a generalization of the incomplete factorial function (incomplete gamma function),[2]

$$I(x,p) = \int_x^{\infty} e^{-u}u^{-p}\,du = \Gamma(1-p, x), \tag{5.176}$$

in which x and p are positive. Again, we seek to evaluate it for large values of x.

Integrating by parts, we obtain

$$\begin{aligned} I(x,p) &= \frac{e^{-x}}{x^p} - p\int_x^{\infty} e^{-u}u^{-p-1}\,du \\ &= \frac{e^{-x}}{x^p} - \frac{pe^{-x}}{x^{p+1}} + p(p+1)\int_x^{\infty} e^{-u}u^{-p-2}\,du. \end{aligned} \tag{5.177}$$

---

[1] This function occurs frequently in astrophysical problems involving gas with a Maxwell–Boltzmann energy distribution.
[2] See also Section 10.5.

Continuing to integrate by parts, we develop the series

$$I(x,p) = e^{-x}\left(\frac{1}{x^p} - \frac{p}{x^{p+1}} + \frac{p(p+1)}{x^{p+2}} - \cdots\right) + (-1)^n \frac{(p+n-1)!}{(p-1)!}\int_x^\infty e^{-u}u^{-p-n}\,du. \tag{5.178}$$

This is a remarkable series. Checking the convergence by the d'Alembert ratio test, we find

$$\begin{aligned}\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|} &= \lim_{n\to\infty}\frac{(p+n)!}{(p+n-1)!}\cdot\frac{1}{x}\\ &= \lim_{n\to\infty}\frac{p+n}{x}\\ &= \infty\end{aligned} \tag{5.179}$$

for all finite values of x. Therefore our series as an infinite series diverges everywhere! Before discarding Eq. 5.178 as worthless, let us see how well a given partial sum approximates the incomplete factorial function, $I(x,p)$.

$$I(x,p) - s_n(x,p) = (-1)^{n+1}\frac{(p+n)!}{(p-1)!}\int_x^\infty e^{-u}u^{-p-n-1}\,du = R_n(x,p). \tag{5.180}$$

In absolute value

$$|I(x,p) - s_n(x,p)| \le \frac{(p+n)!}{(p-1)!}\int_x^\infty e^{-u}u^{-p-n-1}\,du.$$

When we substitute $u = v + x$ the integral becomes

$$\begin{aligned}\int_x^\infty e^{-u}u^{-p-n-1}\,du &= e^{-x}\int_0^\infty e^{-v}(v+x)^{-p-n-1}\,dv\\ &= \frac{e^{-x}}{x^{p+n+1}}\int_0^\infty e^{-v}\left(1+\frac{v}{x}\right)^{-p-n-1}dv.\end{aligned}$$

For large x the final integral approaches 1 and

$$|I(x,p) - s_n(x,p)| \approx \frac{(p+n)!}{(p-1)!}\cdot\frac{e^{-x}}{x^{p+n+1}}. \tag{5.181}$$

This means that if we take x large enough, our partial sum s_n is an arbitrarily good approximation to the desired function $I(x,p)$. Our divergent series (Eq. 5.178) therefore is perfectly good for computations. For this reason it is sometimes called a semiconvergent series. Note that the power of x in the denominator of the remainder $(p+n+1)$ is higher than the power of x in the last term included in $s_n(x,p)$, $(p+n)$.

Since the remainder $R_n(x,p)$ alternates in sign, the successive partial sums give alternately upper and lower bounds for $I(x,p)$. The behavior of the series (with

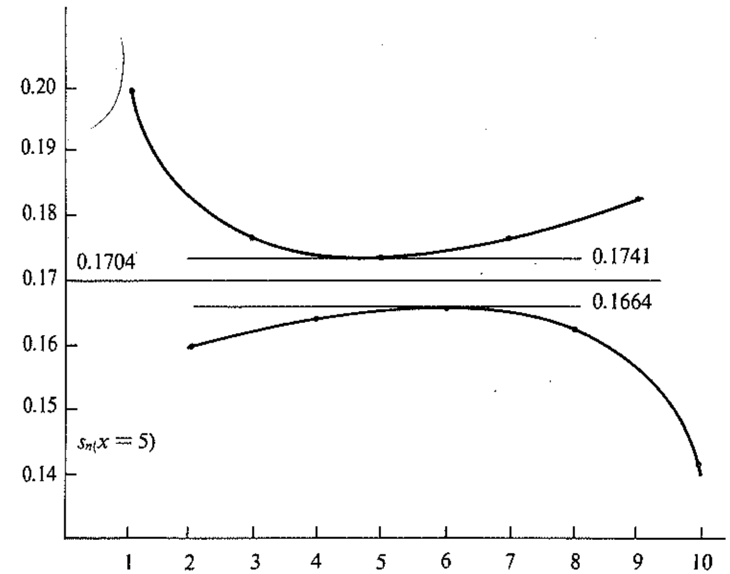


FIG. 5.11 Partial sums of $e^xE_1(x)|_{x=5}$

$p = 1$) as a function of the number of terms included is shown in Fig. 5.11. We have

$$\begin{aligned}e^xE_1(x) &= e^x\int_x^\infty \frac{e^{-u}}{u}\,du\\ &= \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots,\end{aligned} \tag{5.182}$$

which is evaluated at $x = 5$. For a given value of x the successive upper and lower bounds given by the partial sums first converge and then diverge. The optimum determination of $e^xE_1(x)$ is then given by the closest approach of the upper and lower bounds, that is, between $s_4 = s_6 = 0.1664$ and $s_5 = 0.1741$ for $x = 5$. Therefore

$$0.1664 \le e^xE_1(x)|_{x=5} \le 0.1741. \tag{5.183}$$

Actually, from tables,

$$e^xE_1(x)|_{x=5} = 0.1704, \tag{5.184}$$

within the limits established by our asymptotic expansion. Note carefully that inclusion of additional terms in the series expansion beyond the optimum point literally reduces the accuracy of the representation.

As x is increased, the spread between the lowest upper bound and the highest lower bound will diminish. By taking x large enough, one may compute $e^xE_1(x)$ to any desired degree of accuracy. Other properties of $E_1(x)$ are derived and discussed in Section 10.5.

Cosine and Sine Integrals

Asymptotic series may also be developed from definite integrals—if the integrand has the required behavior. As an example, the cosine and sine integrals (Section 10.5) are defined by

$$Ci(x) = -\int_x^\infty \frac{\cos t}{t}\,dt, \tag{5.185}$$

$$si(x) = -\int_x^\infty \frac{\sin t}{t}\,dt. \tag{5.186}$$

Combining these with regular trigonometric functions, we may define

$$\begin{aligned} f(x) &= Ci(x)\sin x - si(x)\cos x = \int_0^\infty \frac{\sin y}{y+x}\,dy, \\ g(x) &= -Ci(x)\cos x - si(x)\sin x = \int_0^\infty \frac{\cos y}{y+x}\,dy, \end{aligned} \tag{5.187}$$

with the new variable $y = t - x$. Going to complex variables, Section 6.1, we have

$$\begin{aligned} g(x) + if(x) &= \int_0^\infty \frac{e^{iy}}{y+x}\,dy \\ &= \int_0^\infty \frac{ie^{-xu}}{1+iu}\,du, \end{aligned} \tag{5.188}$$

in which $u = -iy/x$. The limits of integration, 0 to ∞, rather than 0 to $-i\infty$, may be justified by Cauchy's theorem, Section 6.3. Rationalizing the denominator and equating real part to real part and imaginary part to imaginary part, we obtain

$$\begin{aligned} g(x) &= \int_0^\infty \frac{ue^{-xu}}{1+u^2}\,du, \\ f(x) &= \int_0^\infty \frac{e^{-xu}}{1+u^2}\,du. \end{aligned} \tag{5.189}$$

For convergence of the integrals we must require that $\mathscr{R}(x) > 0$.[3]

Now, to develop the asymptotic expansions, let $v = xu$ and expand the factor $[1 + (v/x)^2]^{-1}$ by the binomial theorem.[4] We have

$$\begin{aligned} f(x) &\approx \frac{1}{x}\int_0^\infty e^{-v} \sum_{n=0}^\infty (-1)^n \frac{v^{2n}}{x^{2n}}\,dv = \frac{1}{x}\sum_{n=0}^\infty (-1)^n \frac{(2n)!}{x^{2n}} \\ g(x) &\approx \frac{1}{x^2}\int_0^\infty e^{-v} \sum_{n=0}^\infty (-1)^n \frac{v^{2n+1}}{x^{2n}}\,dv = \frac{1}{x^2}\sum_{n=0}^\infty (-1)^n \frac{(2n+1)!}{x^{2n}}. \end{aligned} \tag{5.190}$$

[3] $\mathscr{R}(x)$ = real part of (complex) x (compare Section 6.1).

[4] This step is valid for $v \le x$. The contributions from $v \ge x$ will be negligible (for large x) because of the negative exponential. It is because the binomial expansion does not converge for $v \ge x$ that our final series is asymptotic rather than convergent.

From Eqs. 5.187 and 5.190

$$\begin{aligned} Ci(x) &\approx \frac{\sin x}{x}\sum_{n=0}^\infty (-1)^n \frac{(2n)!}{x^{2n}} - \frac{\cos x}{x^2}\sum_{n=0}^\infty (-1)^n \frac{(2n+1)!}{x^{2n}} \\ si(x) &\approx -\frac{\cos x}{x}\sum_{n=0}^\infty (-1)^n \frac{(2n)!}{x^{2n}} - \frac{\sin x}{x^2}\sum_{n=0}^\infty (-1)^n \frac{(2n+1)!}{x^{2n}}, \end{aligned} \tag{5.191}$$

the desired asymptotic expansions.

This technique of expanding the integrand of a definite integral and integrating term by term is applied in Section 11.6 to develop an asymptotic expansion of the modified Bessel function K_ν and in Section 13.6 for expansions of the two confluent hypergeometric functions $M(a,c;x)$ and $U(a,c;x)$.

Definition of Asymptotic Series

The behavior of these series (Eqs. 5.178 and 5.191) is consistent with the defining properties of an asymptotic series.[5] Following Poincaré, we take[6]

$$x^n R_n(x) = x^n[f(x) - s_n(x)],$$

where

$$s_n(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}. \tag{5.193}$$

The asymptotic expansion of $f(x)$ has the properties that

$$\lim_{x\to\infty} x^n R_n(x) = 0, \qquad \text{for fixed } n, \tag{5.194}$$

and

$$\lim_{n\to\infty} x^n R_n(x) = \infty, \qquad \text{for fixed } x.^7 \tag{5.195}$$

For power series, as assumed in the form of $s_n(x)$, $R_n(x) \sim x^{-n-1}$. With conditions (5.194) and (5.195) satisfied, we write

$$f(x) \approx \sum_{n=0}^\infty a_n x^{-n}. \tag{5.196}$$

Note the use of $\approx$ in place of $=$. The function $f(x)$ is equal to the series only in the limit as $x \to \infty$.

[5] It is not necessary that the asymptotic series be a power series. The required property is that the remainder $R_n(x)$ be of higher order than the last term kept—as in Eq. 5.194.

[6] Poincaré's definition allows (or neglects) exponentially decreasing functions. The refinement of Poincaré's definition is of considerable importance for the advanced theory of asymptotic expansions, particularly for extensions into the complex plane. However, for purposes of an introductory treatment and especially for numerical computation with x real and positive, Poincaré's approach is perfectly satisfactory.

[7] This excludes convergent series of inverse powers of x. Some writers feel that this distribution, this exclusion, is artificial and unnecessary.

Asymptotic expansions of two functions may be multiplied together and the result will be an asymptotic expansion of the product of the two functions.

The asymptotic expansion of a given function $f(t)$ may be integrated term by term (just as in a uniformly convergent series of continuous functions) from $x \le t < \infty$ and the result will be an asymptotic expansion of $\int_x^\infty f(t)\,dt$. Term-by-term differentiation, however, is valid only under very special conditions.

Some functions do not possess an asymptotic expansion; e^x is an example of such a function. However, if a function has an asymptotic expansion, it has only one. The correspondence is not one to one; many functions may have the same asymptotic expansion.

One of the most useful and powerful methods of generating asymptotic expansions, the method of steepest descents, will be developed in Section 7.4. Applications include the derivation of Stirling's formula for the (complete) factorial function (Section 10.3) and the asymptotic forms of the various Bessel functions (Section 11.6). Asymptotic series occur fairly often in mathematical physics. One of the earliest and still important approximation treatments of quantum mechanics, the *WKB* expansion, is an asymptotic series.

Applications to Computing

Asymptotic series are frequently used in the computations of functions by modern high-speed electronic computers. This is the case for the Neumann functions $N_0(x)$ and $N_1(x)$, and the modified Bessel functions $I_n(x)$ and $K_n(x)$. The relevant asymptotic series are given as Eqs. 11.127, 11.134, and 11.136. A further discussion of these functions is included in Section 11.6. The asymptotic series for the exponential integral, Eq. 5.182, for the Fresnel integrals, Exercise 5.10.2, and for the Gauss error function, Exercise 5.10.4, are used for the evaluation of these integrals for large values of the argument. How large the argument should be depends on what accuracy is required. In actual practice, a finite portion of the asymptotic series is telescoped by using Chebyshev techniques to optimize the accuracy as discussed in Section 13.4.

EXERCISES

5.10.1 Stirling's formula for the logarithm of the factorial function is

$$\ln(x!) = \frac{1}{2}\ln 2\pi + \left(x + \frac{1}{2}\right)\ln x - x - \sum_{n=1} \frac{B_{2n}}{(2n)(2n-1)} x^{1-2n}.$$

The B_{2n} are the Bernoulli numbers (Section 5.9). Show that Stirling's formula is an *asymptotic* expansion.

5.10.2 Integrating by parts, develop asymptotic expansions of the Fresnel integrals.

(a) $C(x) = \int_0^x \cos\frac{\pi u^2}{2}\,du$

(b) $S(x) = \int_0^x \sin\frac{\pi u^2}{2}\,du.$

These integrals appear in the analysis of a knife-edge diffraction pattern.

5.10.3 Rederive the asymptotic expansions of $Ci(x)$ and $si(x)$ by repeated integration by parts.

Hint. $Ci(x) + i\,si(x) = -\int_x^\infty \frac{e^{it}}{t}\,dt.$

5.10.4 Derive the asymptotic expansion of the Gauss error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$
$$= 1 - \frac{e^{-x^2}}{\sqrt{\pi}x}\left(1 - \frac{1}{2x^2} + \frac{1\cdot 3}{2^2x^4} - \frac{1\cdot 3\cdot 5}{2^3x^6} + \cdots\right).$$

Hint. $\operatorname{erf}(x) = 1 - \operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt.$

Normalized so that $\operatorname{erf}(\infty) = 1$, this function plays an important role in probability theory. It may be expressed in terms of the Fresnel integrals (Exercise 5.10.2), the incomplete gamma functions (Section 10.5), and the confluent hypergeometric functions (Section 13.6).

5.10.5 The asymptotic expressions for the various Bessel functions, Section 11.6, contain the series

$$P_\nu(z) \sim 1 + \sum_{n=1}^\infty (-1)^n \frac{\prod_{s=1}^{2n}[4\nu^2 - (2s-1)^2]}{(2n)!(8z)^{2n}},$$
$$Q_\nu(z) \sim \sum_{n=1}^\infty (-1)^{n+1} \frac{\prod_{s=1}^{2n-1}[\nu^2 - (2s-1)^2]}{(2n-1)!(8z)^{2n-1}}.$$

Show that these two series are indeed asymptotic series.

5.10.6 For $x > 1$

$$\frac{1}{1+x} = \sum_{n=0}^\infty (-1)^n \frac{1}{x^n}.$$

Test this series to see if it is an asymptotic series.

5.10.7 In Exercise 5.9.17 the Euler–Mascheroni constant γ is expressed with a Bernoulli number series:

$$\sum_{k=1} \frac{B_{2k}}{(2k)n^{2k}}.$$

Show that this is an *asymptotic* series.

5.10.8 Develop an asymptotic series for

$$\int_0^\infty e^{-xv}(1+v^2)^{-2}\,dv.$$

Take x to be real and positive.

ANS. $\frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} \cdots \frac{(-1)^n(2n)!}{x^{2n+1}}.$

5.10.9 Calculate partial sums of $e^xE_1(x)$ for $x = 5$, 10, and 15 to exhibit the behavior shown in Fig. 5.11. Determine the width of the throat for $x = 10$ and 15 analogous to Eq. 5.183.

ANS. Throat width: $n = 10$, 0.000051
$n = 15$, 0.0000002.

5.10.10 The knife-edge diffraction pattern is described by

$$I = 0.5I_0\{[C(u_0) + 0.5]^2 + [S(u_0) + 0.5]^2\},$$

where $C(u_0)$ and $S(u_0)$ are the Fresnel integrals. Here I_0 is the incident intensity and I the diffracted intensity. u_0 is proportional to the distance away from the knife edge (measured at right angles to the incident beam). Calculate I/I_0 for u_0 varying from -1.0 to $+4.0$ in steps of 0.1. Tabulate your results and, if a plotting routine is available, plot them.

Check value. $u_0 = 1.0$, $I/I_0 = 1.259226$.

5.10.11 The Euler–Maclaurin integration formula of Section 5.9 provides a way of calculating the Euler–Mascheroni constant γ to high accuracy. Using $f(x) = 1/x$ in Eq. 5.168*b* (with interval $[1, n]$) and the definition of γ, Eq. 5.28, we obtain

$$\gamma = \sum_{s=1}^{n} s^{-1} - \ln n - \frac{1}{2n} + \sum_{k=1} \frac{B_{2k}}{(2k)n^{2k}}.$$

Using double precision arithmetic, calculate γ.
Note. Knuth, D. E., "Euler's constant to 1271 places," *Math. Computation* **16**, 275 (1962). An even more precise calculation appears in Exercise 10.5.16.

ANS. For $n = 1000$,
$\gamma = 0.5772\,1566\,4901$.

5.11 INFINITE PRODUCTS

Consider a succession of positive factors $f_1 \cdot f_2 \cdot f_3 \cdot f_4 \cdots f_n$, $(f_i > 0)$. Using capital pi to indicate product, as capital sigma indicates a sum, we have

$$f_1 \cdot f_2 \cdot f_3 \cdots f_n = \prod_{i=1}^{n} f_i. \tag{5.197}$$

We define p_n, a partial product, in analogy with s_n the partial sum,

$$p_n = \prod_{i=1}^{n} f_i \tag{5.198}$$

and then investigate the limit

$$\lim_{n\to\infty} p_n = P. \tag{5.199}$$

If P is finite (but not zero), we say the infinite product is convergent. If P is infinite *or zero*, the infinite product is labeled divergent.

Since the product will diverge to infinity if

$$\lim_{n=\infty} f_n > 1 \tag{5.200}$$

or to zero for

$$\lim_{n=\infty} f_n < 1, \qquad (\text{and } > 0), \tag{5.201}$$

it is convenient to write our infinite product as

$$\prod_{n=1}^{\infty} (1 + a_n).$$

The condition $a_n \to 0$ is then a necessary (but not sufficient) condition for convergence.

The infinite product may be related to an infinite series by the obvious method of taking the logarithm

$$\ln \prod_{n=1}^{\infty} (1 + a_n) = \sum_{n=1}^{\infty} \ln(1 + a_n). \tag{5.202}$$

A more useful relationship is stated by the following theorem.

Convergence of Infinite Product

If $0 \le a_n < 1$, the infinite products $\prod_{n=1}^{\infty} (1 + a_n)$ and $\prod_{n=1}^{\infty} (1 - a_n)$ converge if $\sum_{n=1}^{\infty} a_n$ converges and diverge if $\sum_{n=1}^{\infty} a_n$ diverges.

Considering the term $1 + a_n$, we see from Eq. 5.90

$$1 + a_n \le e^{a_n}. \tag{5.203}$$

Therefore for the partial product p_n

$$p_n \le e^{s_n}, \tag{5.204}$$

and, letting $n \to \infty$,

$$\prod_{n=1}^{\infty} (1 + a_n) \le \exp \sum_{n=1}^{\infty} a_n, \tag{5.205}$$

thus establishing an upper bound for the infinite product.

To develop a lower bound, we note that

$$p_n = 1 + \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j + \cdots, \qquad > s_n, \tag{5.206}$$

since $a_i \ge 0$. Hence

$$\prod_{n=1}^{\infty} (1 + a_n) \ge \sum_{n=1}^{\infty} a_n. \tag{5.207}$$

If the infinite sum remains finite, the infinite product will also. If the infinite sum diverges, so will the infinite product.

The case of $\prod(1 - a_n)$ is complicated by the negative signs, but a proof that depends on the foregoing proof may be developed by noting that for $a_n < \frac{1}{2}$ (remember $a_n \to 0$ for convergence)

$$(1 - a_n) \le (1 + a_n)^{-1}$$

and

$$(1 - a_n) \ge (1 + 2a_n)^{-1}. \tag{5.208}$$

Sine, Cosine, and Gamma Functions

The reader will recognize that an nth-order polynomial $P_n(x)$ with n real roots may be written as a product of n factors:

$$P_n(x) = (x - x_1)(x - x_2) \cdots (x - x_n) = \prod_{i=1}^{n} (x - x_i). \tag{5.209}$$

In much the same way we may expect that a function with an infinite number of roots may be written as an infinite product, one factor for each root. This is indeed the case for the trigonometric functions. We have two very useful infinite product representations,

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) \tag{5.210}$$

$$\cos x = \prod_{n=1}^{\infty} \left[1 - \frac{4x^2}{(2n-1)^2\pi^2}\right]. \tag{5.211}$$

The most convenient and perhaps most elegant derivation of these two expressions is by the use of complex variables.[1] By our theorem of convergence, Eqs. 5.210 and 5.211 are convergent for all finite values of x. Specifically, for the infinite product for $\sin x$, $a_n = x^2/n^2\pi^2$,

$$\sum_{n=1}^{\infty} a_n = \frac{x^2}{\pi^2} \sum_{n=1}^{\infty} n^{-2} = \frac{x^2}{\pi^2}\zeta(2) = \frac{x^2}{6} \tag{5.212}$$

by Exercise 5.9.6. The series corresponding to Eq. 5.211 behaves in a similar manner.

Equation 5.210 leads to two interesting results. First, if we set $x = \pi/2$, we obtain

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \left[1 - \frac{1}{(2n)^2}\right] = \frac{\pi}{2} \prod_{n=1}^{\infty} \left[\frac{(2n)^2 - 1}{(2n)^2}\right]. \tag{5.213}$$

Solving for $\pi/2$, we have

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left[\frac{(2n)^2}{(2n-1)(2n+1)}\right] = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdots, \tag{5.214}$$

which is Wallis's famous formula for $\pi/2$.

The second result involves the gamma or factorial function (Section 10.1). One definition of the gamma function is

$$\Gamma(x) = \left[x e^{\gamma x} \prod_{r=1}^{\infty} \left(1 + \frac{x}{r}\right) e^{-x/r}\right]^{-1}, \tag{5.215}$$

where γ is the usual Euler–Mascheroni constant (compare Section 5.2). If we take the product of $\Gamma(x)$ and $\Gamma(-x)$, Eq. 5.215 leads to

[1] The derivation appears in *Mathematical Methods for Physicists*, 1st and 2nd eds. (Section 7.3). As an alternative Eq. 5.210 can be obtained from the Weierstrass factorization theorem.

$$\Gamma(x)\Gamma(-x) = -\left[x e^{\gamma x} \prod_{r=1}^{\infty} \left(1 + \frac{x}{r}\right) e^{-x/r} x e^{-\gamma x} \prod \left(1 - \frac{x}{r}\right) e^{x/r}\right]^{-1} = -\left[x^2 \prod_{r=1}^{\infty} \left(1 - \frac{x^2}{r^2}\right)\right]^{-1}. \tag{5.216}$$

Using Eq. 5.210 with x replaced by πx, we obtain

$$\Gamma(x)\Gamma(-x) = -\frac{\pi}{x \sin \pi x}. \tag{5.217}$$

Anticipating a recurrence relation developed in Section 10.1, we have $-x\Gamma(-x) = \Gamma(1-x)$. Eq. 5.217 may be written as

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{5.218}$$

This will be useful in treating the gamma function (Chapter 10).

Strictly speaking, we should check the range of x for which Eq. 5.215 is convergent. Clearly, individual factors will vanish for $x = 0, -1, -2, \ldots$. The proof that the infinite product converges for all other (finite) values of x is left as Exercise 5.11.9.

These infinite products have a variety of uses in analytical mathematics. However, because of rather slow convergence, they are not suitable for precise numerical work.

EXERCISES

5.11.1 Using

$$\ln \prod_{n=1}^{\infty} (1 \pm a_n) = \sum_{n=1}^{\infty} \ln(1 \pm a_n)$$

and the Maclaurin expansion of $\ln(1 \pm a_n)$, show that the infinite product $\prod_{n=1}^{\infty}(1 \pm a_n)$ converges or diverges with the infinite series $\sum_{n=1}^{\infty} a_n$.

5.11.2 An infinite product appears in the form

$$\prod_{n=1}^{\infty} \left(\frac{1 + a/n}{1 + b/n}\right),$$

where a and b are constants. Show that this infinite product converges only if $a = b$.

5.11.3 Show that the infinite product representations of $\sin x$ and $\cos x$ are consistent with the identity $2 \sin x \cos x = \sin 2x$.

5.11.4 Determine the limit to which

$$\prod_{n=2}^{\infty} \left(1 + \frac{(-1)^n}{n}\right)$$

converges.

5.11.5 Show that

$$\prod_{n=2}^{\infty}\left[1-\frac{2}{n(n+1)}\right]=\frac{1}{3}.$$

5.11.6 Prove that

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}.$$

5.11.7 Using the infinite product representations of $\sin x$, show that

$$x\cot x = 1 - 2\sum_{m,n=1}^{\infty}\left(\frac{x}{n\pi}\right)^{2m},$$

hence that the Bernoulli number

$$B_{2n} = (-1)^{n-1}\frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n),$$

5.11.8 Verify the Euler identity

$$\prod_{p=1}^{\infty}(1+z^p)=\prod_{q=1}^{\infty}(1-z^{2q-1})^{-1}, \qquad |z|<1.$$

5.11.9 Show that $\prod_{r=1}^{\infty}(1+x/r)e^{-x/r}$ converges for all finite x (except for the zeros of $1+x/r$).
Hint. Write the nth factor as $1+a_n$.

5.11.10 Calculate $\cos x$ from its infinite product representation, Eq. 5.211, using (a) 10, (b) 100, and (c) 1000 factors in the product. Calculate the absolute error. Note how slowly the partial products converge—making the infinite product quite unsuitable for precise numerical work.

ANS. For 1000 factors $\cos\pi = -1.00051$.

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