I.2. The Vector Product of Two Vectors

\[ \mathbf{C} = \mathbf{a} \times \mathbf{b} \]

where \( \mathbf{a} \) and \( \mathbf{b} \) are vectors, and \( \times \) denotes the cross product. The vector product of two vectors is used to define several quantities in physics, such as torque, angular momentum, and magnetic force.

The scalar product (or dot product) of two vectors \( \mathbf{a} \) and \( \mathbf{b} \) is defined as:

\[ \mathbf{a} \cdot \mathbf{b} = ab \cos \theta \]

where \( a \) and \( b \) are the magnitudes of the vectors and \( \theta \) is the angle between them.

The following properties of the scalar product are important:

1. (Commutative) \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \)
2. (Distributive) \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \)
3. (Scalar Multiplication) \( (k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}) \)

We next consider a method of combining two vectors—the so-called rotational way of expressing the vector \( \mathbf{a} \).

The following way of expressing the vector \( \mathbf{a} \) is illustrated:

The position vector is introduced, and in this vector the rotational axis is also introduced.

Sometimes we want to describe a vector in terms of its components along the usual unit vectors.

I.11. Unit Vectors

The unit vectors \( \mathbf{i} \) and \( \mathbf{j} \) are defined by:

\[ \mathbf{i} = \mathbf{a} \quad \mathbf{j} = \mathbf{b} \]

where \( \mathbf{a} \) and \( \mathbf{b} \) are vectors. We can use these unit vectors to express any vector as a linear combination of them.

\[ \mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} \]

The unit vectors are orthogonal (because they are perpendicular to each other). Therefore, the scalar product of any two unit vectors (because they are orthogonal) is:

\[ \mathbf{i} \cdot \mathbf{j} = 0 \]

The cosine of the angle between the vectors is:

\[ \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{ab} \]

We have seen (Equation 1.56) that the scalar product of two vectors has a direction.
\[ \mathbf{A} \times \mathbf{B} = \mathbf{C} \]

Since \( \mathbf{B} \times \mathbf{C} = \mathbf{A} \) and \( \mathbf{A} \times \mathbf{C} = \mathbf{B} \), the plane defined by \( \mathbf{A} \) and \( \mathbf{B} \) is perpendicular to the plane defined by \( \mathbf{A} \) and \( \mathbf{C} \), and vice versa.

**Example 1.2**

Let \( \mathbf{A} \) and \( \mathbf{B} \) be two non-zero vectors, and let \( \theta \) be the angle between them. Geometrically, \( \theta \) is the area of the parallelogram defined by the vectors. Geometrically, \( \theta \) is the area of the parallelogram defined by the vectors. Geometrically, \( \theta \) is the area of the parallelogram defined by the vectors.

**Theorem 1.2**

The cross product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is defined as

\[ \mathbf{A} \times \mathbf{B} = \mathbf{C} \]

where \( \mathbf{C} \) is the vector such that

\[ \mathbf{C} = \mathbf{A} \times \mathbf{B} \]

and \( \mathbf{A} \times \mathbf{B} \) is orthogonal to both \( \mathbf{A} \) and \( \mathbf{B} \).

**Proof**

1. **Scalar Product**: \( \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta \)
2. **Vector Product**: \( \mathbf{A} \times \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \sin \theta \mathbf{C} \)

where \( \mathbf{C} \) is the cross product of \( \mathbf{A} \) and \( \mathbf{B} \).

**Example 1.3**

Let \( \mathbf{A} \) and \( \mathbf{B} \) be two non-zero vectors, and let \( \theta \) be the angle between them. Geometrically, \( \theta \) is the area of the parallelogram defined by the vectors. Geometrically, \( \theta \) is the area of the parallelogram defined by the vectors.

**Theorem 1.3**

The dot product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is defined as

\[ \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta \]

where \( \theta \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \).

**Proof**

1. **Scalar Product**: \( \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta \)
2. **Vector Product**: \( \mathbf{A} \times \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \sin \theta \mathbf{C} \)

where \( \mathbf{C} \) is the cross product of \( \mathbf{A} \) and \( \mathbf{B} \).

**Example 1.4**

Let \( \mathbf{A} \) and \( \mathbf{B} \) be two non-zero vectors, and let \( \theta \) be the angle between them. Geometrically, \( \theta \) is the area of the parallelogram defined by the vectors. Geometrically, \( \theta \) is the area of the parallelogram defined by the vectors.

**Theorem 1.4**

The cross product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is defined as

\[ \mathbf{A} \times \mathbf{B} = \mathbf{C} \]

where \( \mathbf{C} \) is the vector such that

\[ \mathbf{C} = \mathbf{A} \times \mathbf{B} \]

and \( \mathbf{A} \times \mathbf{B} \) is orthogonal to both \( \mathbf{A} \) and \( \mathbf{B} \).

**Proof**

1. **Scalar Product**: \( \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta \)
2. **Vector Product**: \( \mathbf{A} \times \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \sin \theta \mathbf{C} \)

where \( \mathbf{C} \) is the cross product of \( \mathbf{A} \) and \( \mathbf{B} \).

**Example 1.5**

Let \( \mathbf{A} \) and \( \mathbf{B} \) be two non-zero vectors, and let \( \theta \) be the angle between them. Geometrically, \( \theta \) is the area of the parallelogram defined by the vectors. Geometrically, \( \theta \) is the area of the parallelogram defined by the vectors.

**Theorem 1.5**

The dot product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is defined as

\[ \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta \]

where \( \theta \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \).

**Proof**

1. **Scalar Product**: \( \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta \)
2. **Vector Product**: \( \mathbf{A} \times \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \sin \theta \mathbf{C} \)

where \( \mathbf{C} \) is the cross product of \( \mathbf{A} \) and \( \mathbf{B} \).

**Example 1.6**

Let \( \mathbf{A} \) and \( \mathbf{B} \) be two non-zero vectors, and let \( \theta \) be the angle between them. Geometrically, \( \theta \) is the area of the parallelogram defined by the vectors. Geometrically, \( \theta \) is the area of the parallelogram defined by the vectors.

**Theorem 1.6**

The cross product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is defined as

\[ \mathbf{A} \times \mathbf{B} = \mathbf{C} \]

where \( \mathbf{C} \) is the vector such that

\[ \mathbf{C} = \mathbf{A} \times \mathbf{B} \]

and \( \mathbf{A} \times \mathbf{B} \) is orthogonal to both \( \mathbf{A} \) and \( \mathbf{B} \).

**Proof**

1. **Scalar Product**: \( \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos \theta \)
2. **Vector Product**: \( \mathbf{A} \times \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \sin \theta \mathbf{C} \)

where \( \mathbf{C} \) is the cross product of \( \mathbf{A} \) and \( \mathbf{B} \).
We state the following identities without proof:

\[
\begin{align*}
\text{(1.8a)} \quad & (\mathbf{c} \times \mathbf{d}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} \\
\text{(1.8b)} \quad & (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} = (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} - (\mathbf{c} \cdot \mathbf{b}) \mathbf{a} \\
\text{(1.8c)} \quad & (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d}) \\
\text{(1.8d)} \quad & \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \\
\text{(1.8e)} \quad & \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \\
\text{(1.8f)} \quad & \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{b} \cdot \mathbf{a}) \mathbf{c} - (\mathbf{b} \cdot \mathbf{c}) \mathbf{a} \\
\text{(1.8g)} \quad & \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b}) \mathbf{a} - (\mathbf{c} \cdot \mathbf{a}) \mathbf{b}
\end{align*}
\]

We therefore have

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})
\]

By direct expansion and comparison with Equation 1.8a, we can verify a determinant expression for the vector product:

\[
\begin{vmatrix}
\mathbf{a} & \mathbf{b} & \mathbf{c} \\
\mathbf{b} & \mathbf{c} & \mathbf{a} \\
\mathbf{c} & \mathbf{a} & \mathbf{b}
\end{vmatrix} = \mathbf{c} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}
\]

The vector product \( \mathbf{c} = \mathbf{a} \times \mathbf{b} \) for example, can now be expressed as

\[
\mathbf{c} = (\mathbf{a} \cdot \mathbf{b}) \mathbf{e}_3 - (\mathbf{a} \cdot \mathbf{b}) \mathbf{e}_1 + (\mathbf{a} \cdot \mathbf{b}) \mathbf{e}_2
\]

We can now use the permutation symbol to express this result as

\[
\mathbf{e}_j = \mathbf{e}_k = \mathbf{e}_l = \mathbf{e}_1 \times \mathbf{e}_2 \times \mathbf{e}_3
\]

The orthogonality of the unit vectors \( \mathbf{e}_j \) requires the vector product to be

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b}
\]

There, finally, since each term in parentheses on the right-hand side is just a scalar product, we have

\[
\begin{align*}
& = (a^2 \mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (a^2 \mathbf{b} \cdot \mathbf{c}) \mathbf{a} \\
& = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{b} \cdot \mathbf{c}) \mathbf{a}
\end{align*}
\]

This equation can be rearranged to obtain

\[
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{b} \cdot \mathbf{c}) \mathbf{a}
\]

Notice that the determination of the vector product of two vectors

\[
\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}
\]

is in general,

\[
\begin{align*}
& \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} \\
& = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b}
\end{align*}
\]

from the definition:

We should note the following properties of the vector product that result

\[
\begin{align*}
& (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} \\
& \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}
\end{align*}
\]

- Under a proper rotation,
- The magnitude of the vector product is now complete.

The direction of the vector product is always perpendicular to the plane of

\[
\mathbf{a} \times \mathbf{b}
\]

Furthermore, the direction of the vector product is always perpendicular to the plane of

\[
\mathbf{a} \times \mathbf{b}
\]

The positive direction of the vector product is chosen to be the direction of advance of a vector normal to the plane and of magnitude equal to the area C is evidently such.

Example 16

\[
\begin{align*}
& (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b}
\end{align*}
\]

Another important result (see Problem 1.27) is

\[
\begin{align*}
& \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}
\end{align*}
\]

Example 17

\[
\begin{align*}
& \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \cdot \mathbf{c}) \mathbf{b}
\end{align*}
\]