Problem 1

Consider the chiral Lagrangian for $SU_L(3) \times SU_R(3)$:

$$\mathcal{L} = \frac{f^2}{8} \text{Tr}[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma] + \frac{B f^2}{4} \text{Tr}[m_q \Sigma + m_q \Sigma^\dagger].$$

The matrix of Goldstone bosons, $\Sigma$, is

$$\Sigma = e^{2iM/f} \quad \text{where} \quad M = \begin{pmatrix} \pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} & K^0 \\ K^- & K^0 & -2\eta/\sqrt{6} \end{pmatrix},$$

$m_q = \text{diag}(m_u, m_d, m_s)$ and $f = f_\pi = 130$ MeV.

a) Take the isospin limit, $m_u = m_d$, and calculate the masses of $\pi$, $K$, and $\eta$ in terms of the quark masses and the parameter $B$.

b) Verify the Gell-Mann-Okubo relation: $3m_\eta^2 = 4m_K^2 - m_\pi^2$.

c) Calculate $\hat{m}/m_s$, where $\hat{m} = (m_u + m_d)/2$, in terms of $m_\pi$ and $m_K$ and obtain a numerical estimate using $m_\pi = 138$ MeV and $m_K = 496$ MeV.

Problem 2

a) For QCD with two flavors of quarks, $u$ and $d$, construct the Noether current, $J^\mu_{3A}$, for the symmetry transformation:

$$\delta u = i\gamma_5 \alpha u \quad \delta d = -i\gamma_5 \alpha d. \quad (1)$$

b) Calculate the coefficient in the $J^\mu_{3A}$ anomaly

$$\partial_\mu J^\mu_{3A} = -C \frac{e^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}.$$

c) Now consider a low energy effective theory of Goldstone bosons, $\pi^0$ and $\pi^\pm$. Show that under the transformation in Eq. (1) $\pi^\pm$ are invariant while $\pi^0$ transform by a shift proportional to $\alpha$. Hint: the effective theory is expressed in terms of fields

$$\Sigma = e^{2iM/f_\pi} \quad \text{where} \quad M = \begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}.$$ 

Under $SU_L(2) \times SU_R(2)$, $\Sigma \rightarrow L \Sigma R^\dagger$. What $L$ and $R$ correspond to Eq. (1)?

d) Because the Noether current corresponding to Eq. (1) is nonconserved, the action must vary as

$$\delta S = \int d^4 x \alpha \partial_\mu J^\mu_{3A},$$
under the transformation generated by Eq. (1). Use this and your result from part c) to argue that there must be a term in the effective Lagrangian of the form
\[ \mathcal{L} = g \pi^0 F_{\mu\nu} \tilde{F}^{\mu\nu}, \] (2)
and determine \( g \). Calculate the decay rate \( \pi^0 \to \gamma\gamma \).

**Problem 3**

Show that baryon number (quark number divided by three), \( B \), and lepton number, \( L \), are classically conserved but are not symmetries of the Standard Model due to anomalies. Right down the anomaly equations for the baryon and lepton number currents. Show that \( B - L \) has no anomaly.

**Problem 4**

If QCD has a nonvanishing \( \theta \) term, its CP violating effects on mesons can be calculated using the Lagrangian in Problem 1. The trick is to note that under the \( U(1) \) chiral transformations
\[ q_{Li} \to e^{i\alpha_i} q_{Li}, \quad q_{Ri} \to e^{-i\alpha_i} q_{Ri}, \] (3)
where \( i = u, d, s \) is the quark flavor, the \( \theta \) parameter transforms as \( \theta \to \theta - 2 \sum_{i=u,d,s} \alpha_i \), so the \( \theta \) term can be rotated away in exchange for a complex mass matrix. The complex mass matrix then enters the chiral Lagrangian in Eq. (1) and induces CP violating effects.

a) Show that the transformation in Eq. (3) modifies the quark mass matrix as
\[
\text{diag}(m_u, m_d, m_s) \to \text{diag}(e^{-2i\alpha_u} m_u, e^{-2i\alpha_d} m_d, e^{-2i\alpha_s} m_s)
\approx \text{diag}(m_u, m_d, m_s) - 2i \text{diag}(m_u \alpha_u, m_d \alpha_d, m_s \alpha_s) \\
\equiv \text{diag}(m_u, m_d, m_s) - i\delta m_q.
\]
Here we have expanded in the \( \alpha_i \) because we expect \( \theta \) to be small.

b) Show that the modification to the chiral lagrangian in Eq. (1) is
\[ \mathcal{L}' = -i B f^2 \frac{1}{4} \text{Tr}[\delta m_q \Sigma - \delta m_q \Sigma^\dagger ] \] (4)

c) Clearly we want \( 2 \sum_{i=u,d,s} \alpha_i = \theta \) to rotate away the \( \theta \) term. The remaining freedom in choosing the \( \alpha_i \) is removed by requiring
\[ \delta m_q = \lambda \mathbb{1}, \]
where \( \mathbb{1} \) is the identity matrix. Why do we choose to do this? (Hint: when are there terms linear in Goldstone bosons in Eq. (4)?) Find \( \lambda \).

d) Use \( \mathcal{L}' \) to calculate the decay rate for \( \eta \to \pi\pi \). Explain why this violates CP.