

~~Suppose~~ Non Abelian Anomalies

so far we have studied anomaly for axial current in QED. we found:

$$\partial_\mu J_V^\mu = 0 \quad \partial_\mu J_A^\mu = -\frac{e^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

often $J_V^\mu = \bar{\psi} \gamma^\mu \psi$, $J_A^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi$

often in what follows we will be interested in anomalies in chiral currents.

$$J_L^\mu = \frac{1}{2} (J_V^\mu + J_A^\mu) =$$

$$J_R^\mu = \frac{1}{2} (J_V^\mu - J_A^\mu)$$

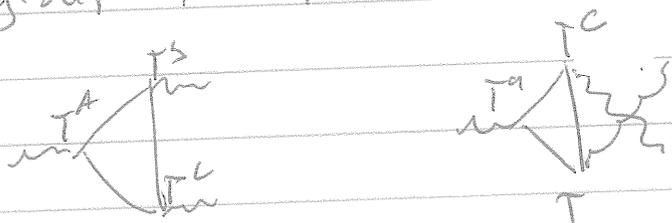
$$\partial_\mu J_L^\mu = \frac{e^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = \frac{e^2}{16\pi^2} \tilde{F}^{\mu\nu} F_{\mu\nu}$$

$$\partial_\mu J_R^\mu = -\frac{e^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = -\frac{e^2}{16\pi^2} F^{\mu\nu} \tilde{F}_{\mu\nu}$$

Now suppose $J_{L,R}^\mu$ are symmetry generator of a non-abelian group.

$$J_L^\mu = J_L^{a\mu} T^a$$

The triangle diagrams are now same as before, but there is an additional group theory factor



~~Therefore~~ let's focus on left-handed chiral currents:

$$\partial_\mu j_\nu^{\mu A} = -\frac{1}{32\pi^2} d_L^{ABC} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^B F_{\alpha\beta}^C$$

where $d_L^{ABC} = \text{tr} \left[\left\{ t_L^A, t_L^B \right\} t_L^C \right]$

$$d_R^{ABC} = -\frac{1}{2} \text{tr} \left[\left\{ t_R^A, t_R^B \right\} t_R^C \right]$$

If non-abelian $J_\mu^A T^A$ are global currents, then this means some global symmetries of the classical action will not be respected by quantum theory. ~~For~~

If non-abelian $J_\mu^A T^A$ are coupled to gauge fields as in the standard model then chiral anomaly presents a problem - theory is inconsistent we must arrange for the anomalies to cancel.

when do anomalies cancel:

or pseudoreal

1) Real representations: $t_R^A = U(-t_A^{RT})U^{-1} = U t_R^A U^{-1}$

ie. irrep R is related to irrep \bar{R} by similarity transformation. Then

$$\begin{aligned} \text{Tr} \left[\begin{matrix} A & B & C \\ t_R^A & t_R^B & t_R^C \end{matrix} \right] &= \text{Tr} \left[\begin{matrix} -t_R^{AT} & -t_R^{BT} \\ -t_R^C \end{matrix} \right] \\ &= -\text{Tr} \left[t_R^C \begin{matrix} t_R^A & t_R^B \end{matrix} \right] \\ &= -\text{Tr} \left[\begin{matrix} t_R^A & t_R^B \\ t_R^C \end{matrix} \right] \end{aligned}$$

no anomalies when fermions are in ~~real~~ real rep.

2) vector-like theories $J^{AA} = \bar{\psi} \gamma^m T^a \psi$
 $= \bar{\psi}_L \gamma^m T^a \psi_L + \bar{\psi}_R \gamma^m T^a \psi_R$

since L, R fields give equal and opposite contribution to anomaly, this is anomaly free.

This can be thought of as a special case of 1) since we can express all fields as left-handed spinors

$$\bar{\psi}_L = \begin{pmatrix} \psi_L \\ c\psi_R^* \end{pmatrix} \begin{matrix} R & \text{irrep} \\ R & \text{irrep} \end{matrix}$$

another special case of (i)

$$SU(2) \quad t^a = \frac{\sigma^a}{2} \quad \sigma^{AB} = \frac{1}{2} \epsilon^{ABC} \sigma^C$$

$$d^{ABC} = \frac{1}{2} \text{tr} \left[\left\{ t^A, t^B \right\} t^C \right] \propto \frac{1}{2} \text{tr} \left[t^C \right] = 0$$

$SU(2)$ is ~~not~~ ^{also} pseudo real

Algebras w/ only real or pseudo real irreps

$$SO(2n+1), SO(4n) \quad Sp(2n) \quad (G_2, F_4, E_7, E_8)$$

$n \geq 2 \quad n \geq 3$

Some groups have vanishing d^{ABC} des.

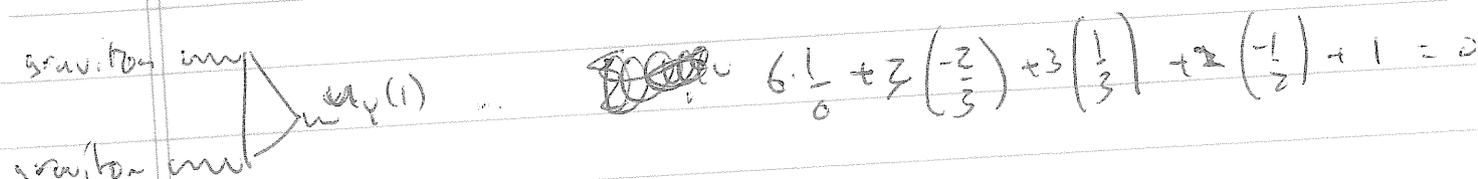
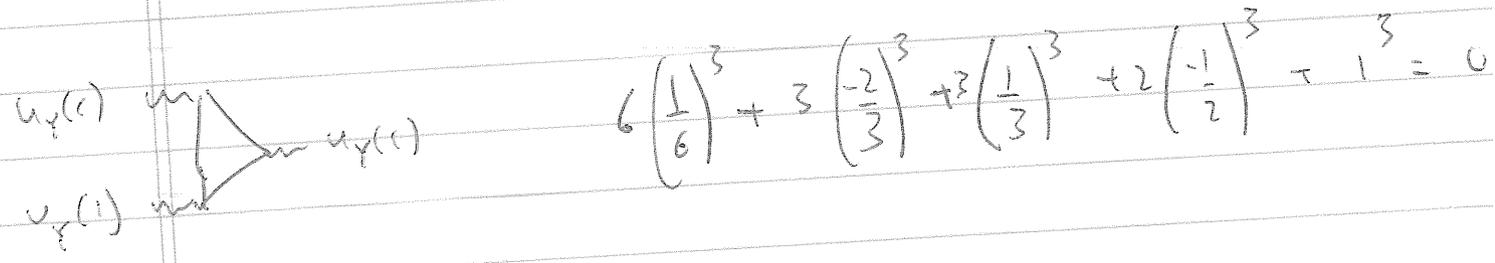
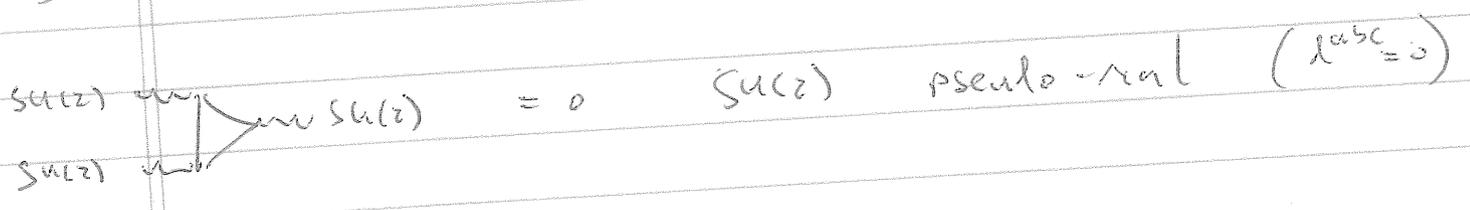
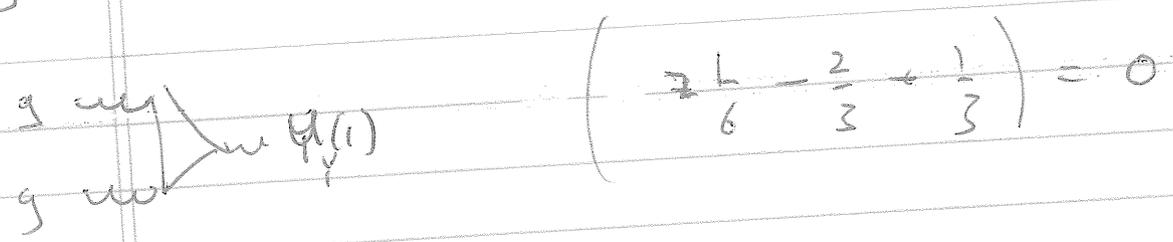
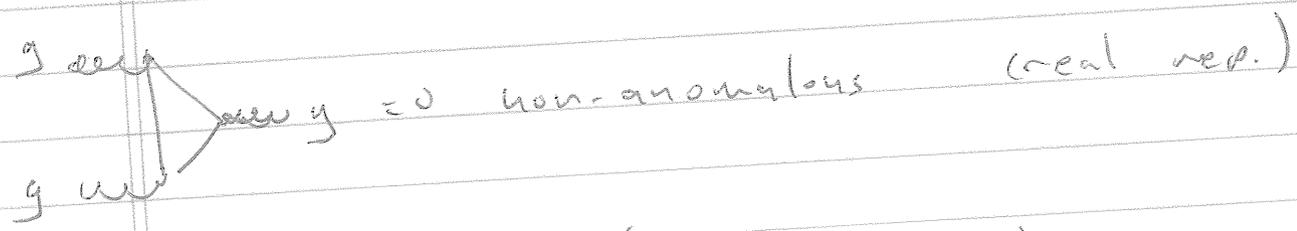
$$SO(4n+2) \quad n \geq 2 \quad \left(\begin{array}{l} SO(2) \sim U(1) \\ SO(6) \sim SU(4) \end{array} \right. \begin{array}{l} \text{have} \\ \text{complex} \end{array} \right)$$

E_6

\Rightarrow Anomalies are only possible if: $SU(n \geq 3), U(1)$

Anomaly cancellation in SM

	<u>SU(3)</u>	<u>SU(2)</u>	<u>U(1)</u>
one generation of left-handed fermions	$\begin{pmatrix} u \\ d \end{pmatrix}_L$	3	$\frac{2}{6}$
	u_R^*	$\bar{3}$	$-\frac{2}{3}$
	d_R^*	$\bar{3}$	$\frac{1}{3}$
	$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	1	$-\frac{1}{2}$
	e_R^*	1	+1



an interesting way to interpret anomaly cancellation is to note that we can add

$$\chi_{\mathbb{R}} \quad \frac{su(3)}{1} \quad \frac{su(2)}{1} \quad \frac{su(1)}{0}$$

this is a gauge singlet and will not contribute to anomalies.

16 states

- 6 $\begin{pmatrix} u \\ d \end{pmatrix}_L$ spinor irrep
- 3 u_R^c
- 3 d_R^c 16 - of $so(10)$
- 3 $\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$ + $so(10)$ is anomaly free!
- 1 e_R^c
- 1 ν_R

spinor irreps of $\{P^a, P^b\} = \delta^{ab} \quad a=1, \dots, 10$

~~definition~~ $\gamma_{\mu} = P^0 \pm i P^1$ $\{a_+, a_+\} = 0 = \{a_-, a_-\}$

$\{a_+, a_-\} = 1$

algebra of fermion oscillator

a_{\pm} clearly have five of these $\Rightarrow 32$ irreducible representation

$32 \rightarrow 10 \times 16$ distinguished by ev. order

$$P^{10} = P^0 P^1 P^2 P^3 P^4 \dots P^9$$

Spontaneous Chiral Symmetry Breaking in QCD

$$\begin{aligned} \mathcal{L}_{\text{QCD}} &= -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} + \sum_i \bar{q}_i (i \not{D} - m_i) q_i \\ &= -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} + \sum_L \bar{q}_L^i i \not{D} q_L^i + \sum_R \bar{q}_R^i i \not{D} q_R^i \\ &\quad - \sum_i m_i (\bar{q}_L^i q_R^i + \bar{q}_R^i q_L^i) \end{aligned}$$

We will consider QCD w/ N light quarks (micro QCD)
 $N=2$ (u, d)
 $N=3$ (u, d, s)

Chiral symmetry $q_L \rightarrow L q_L$ $q_R \rightarrow R q_R$

~~Global symmetry~~ $L \in SU_L(N)$ $R \in SU_R(N)$

Symmetries: $U(1)_{B} \times SU_L(N) \times SU_R(N)$

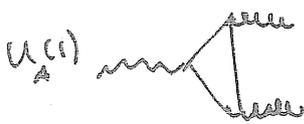
\uparrow baryon number $q_{L,R} \rightarrow e^{i\theta} q_{L,R}$

Non-symmetry: axial $U(1) = \frac{U(1)}{A}$ $q_L \rightarrow e^{i\theta} q_L$ $q_R \rightarrow e^{-i\theta} q_R$

classically $\int d^4x j_5^{\mu} = 0$

quantum mech. $\int d^4x j_5^{\mu} = n_f \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\alpha\beta} G^{\mu\nu} G^{\alpha\beta}$

$$\int d^4x j_5^{\mu} = n_f \frac{g^2}{16\pi^2} G^{\mu\nu} \tilde{G}_{\mu\nu}$$



the axial anomaly

8

Since
$$\begin{aligned}
 \mathcal{L} &= \int d^4x \mathcal{L} \approx \int d^4x \frac{1}{2} \bar{\psi} \not{\partial} \psi \\
 &= \int d^4x \frac{g^2}{16\pi^2} G^{\mu\nu} \tilde{G}_{\mu\nu}
 \end{aligned}$$

so action is not invariant under $U(1)_A$

Chiral symmetry is broken by vacuum condensate

$$\langle \bar{q}^i q^i \rangle = \langle \bar{q}_L^i q_R^i - \bar{q}_R^i q_L^i \rangle$$

in limit $m_i \rightarrow 0$ $\langle \bar{q} q \rangle$ is same for all light quarks. Under chiral transformation

$$\langle \bar{q}_L q_R \rangle \rightarrow \langle \bar{q}_L L^\dagger R q_R \rangle$$

$L^\dagger R$ vacuum invariant under $L=R$ $L^\dagger R = L^{-1} R = 1$

$\Rightarrow L=R$

$$L = e^{i\vec{\theta}_L \cdot \vec{T}_L} \quad R = e^{i\vec{\theta}_R \cdot \vec{T}_R}$$

T^a

generators of $SU(N)$

vector

$$\vec{\theta}_L = \vec{\theta}_R$$

or

$$L=R$$

axial

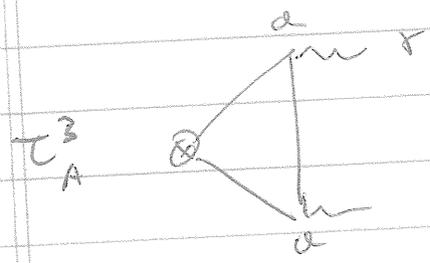
$$\vec{\theta}_L = -\vec{\theta}_R$$

$$L=R^{-1}$$

$$SU_L(N) \times SU_R(N) \rightarrow SU_{L+R}(N) = \underline{SU(N)}$$

chiral symmetry spontaneously broken to vector subgroup.

There is another anomaly in flavor currents when I consider $U(1)_{em}$



$$j^{\mu AB} = \frac{1}{2} \bar{u} \gamma^\mu \gamma_5 u - \frac{1}{2} \bar{d} \gamma^\mu \gamma_5 d$$

$$= \bar{q} \gamma^\mu \gamma_5 \tau_3 q$$

this anomaly is

$$\partial_\mu j^{\mu AB} = -\frac{1}{16\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma} \text{Tr}[\alpha^2 \tau_3]$$

for up quarks

$$\text{Tr}[\alpha^2 \tau_3] = N_c \left[\left(\frac{2}{3}\right)^2 \frac{1}{2} - \left(\frac{-1}{3}\right)^2 \frac{1}{2} \right] e^2$$

$$= 3 \left(\frac{2}{9} - \frac{1}{9} \right) = \frac{3 \cdot 3}{18} = \frac{1}{2}$$

$$\partial_\mu j^{\mu AB} = -\frac{e^2}{16\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}$$

$j^{\mu AB} = \bar{q} \gamma^\mu \gamma_5 \tau_3 q$ the anomaly is

responsible for $\pi^0 \rightarrow \gamma\gamma$

note this current is non-anomalous in pure QCD!

θ term in QCD

(10)

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2} \text{Tr} [G_{\mu\nu} G^{\mu\nu}] - \sum_i \bar{\psi}_i (\not{\partial} - \gamma_i) \psi_i \\ - \theta \frac{g^2}{16\pi^2} \text{Tr} [G_{\mu\nu} \hat{G}^{\mu\nu}]$$

under $U(1)$ $q_L \rightarrow e^{i\alpha} q_L$
 $\bar{q}_L \rightarrow e^{-i\alpha} \bar{q}_L$

$\theta \rightarrow \theta - 2\pi\alpha$ because of anomaly.

next we'll show this violates CP,
is a total derivative and try
to interpret meaning of this term.

~~$$F_{\mu\nu} F^{\mu\nu} = 2(\vec{E}^2 - \vec{B}^2)$$~~

$$F_{\mu\nu} F^{\mu\nu} = -2(\vec{E}^2 - \vec{B}^2)$$

$$F^{\mu\nu} \tilde{F}_{\mu\nu} = 4\vec{E} \cdot \vec{B}$$

under P: $\vec{E} \rightarrow -\vec{E}$ $\vec{B} \rightarrow \vec{B}$ $F^{\mu\nu} \tilde{F}_{\mu\nu}$ P odd

T: $\vec{E} \rightarrow \vec{E}$ $\vec{B} \rightarrow -\vec{B}$

so θ term violates P, CP (or T, assuming CPT)

therefore must be very small since strong interaction preserve P, CP (as far as we know.) Later will do ChPT estimate of neutron electric dipole moment

$\theta < 10^{-8}$ why? strong CP problem.

Another important point: $F^{\mu\nu} \tilde{F}_{\mu\nu}$ is a total derivative:

QED

$$\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$$= 4 \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma$$

$$= 4 \epsilon^{\mu\nu\rho\sigma} \partial_\mu [A_\nu \partial_\rho A_\sigma]$$

since $\epsilon^{\mu\nu\rho\sigma} \partial_\nu \partial_\rho A_\sigma = 0$

this means contribution to action is a surface term. Does not contribute to e.o.m., Feynman diagrams

Θ -term is also total derivative in ~~non Abelian~~ ⁽¹²⁾ YM theory

$$k^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta} \text{Tr} \left[A_\nu \partial_\alpha A_\beta + i\frac{2}{3} A_\nu A_\alpha A_\beta \right] \quad A_\mu = A_\mu^a T^a$$

Computing $\partial_\mu k^{\mu\nu}$:

$$\begin{aligned} \partial_\mu \epsilon^{\mu\nu\alpha\beta} \text{Tr} [A_\nu \partial_\alpha A_\beta] &= \text{Tr} \epsilon^{\mu\nu\alpha\beta} \text{Tr} [\partial_\mu A_\nu \partial_\alpha A_\beta] \\ &= \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \text{Tr} [\partial_{[\mu} A_{\nu]} \partial_{[\alpha} A_{\beta]}] \end{aligned}$$

$$\partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\begin{aligned} & \frac{2}{3} \text{Tr} [\partial_\mu A_\nu A_\alpha A_\beta + A_\nu \partial_\mu A_\alpha A_\beta + A_\nu A_\alpha \partial_\mu A_\beta] \quad \times \epsilon^{\mu\nu\alpha\beta} \\ & - \frac{2}{3} \text{Tr} [\partial_\mu A_\nu A_\alpha A_\beta - A_\alpha \partial_\mu A_\nu A_\beta + A_\beta A_\alpha \partial_\mu A_\nu] \quad \times \epsilon^{\mu\nu\alpha\beta} \\ & = \frac{2}{3} \text{Tr} [\partial_\mu A_\nu A_\alpha A_\beta + \partial_\mu A_\alpha A_\nu A_\beta + \partial_\mu A_\nu A_\alpha A_\beta] \quad \times \epsilon^{\mu\nu\alpha\beta} \\ & = 2 \text{Tr} [\partial_\mu A_\nu A_\alpha A_\beta] = \frac{1}{2} \text{Tr} [\partial_{[\mu} A_{\nu]} A_{[\alpha} A_{\beta]}] \quad \times \epsilon^{\mu\nu\alpha\beta} \\ & = \frac{1}{4} \text{Tr} [\partial_{[\mu} A_{\nu]} A_{[\alpha} A_{\beta]}] + \frac{1}{4} \text{Tr} [\partial_{[\alpha} A_{\beta]} A_{[\mu} A_{\nu]}] \quad \times \epsilon^{\mu\nu\alpha\beta} \end{aligned}$$

$$\epsilon^{\mu\nu\alpha\beta} \text{Tr} [A_\mu, A_\nu] [A_\alpha, A_\beta] = \epsilon^{\mu\nu\alpha\beta} \text{Tr} [A_\mu A_\nu (A_\alpha, A_\beta) - A_\nu A_\mu (A_\alpha, A_\beta)]$$

$$= \epsilon^{\mu\nu\alpha\beta} \text{Tr} [A_\mu A_\nu (A_\alpha, A_\beta) - (A_\alpha, A_\beta) A_\nu A_\mu]$$

$$= \epsilon^{\mu\nu\alpha\beta} \text{Tr} [A_\mu [A_\nu, [A_\alpha, A_\beta]]]$$

= 0 due to Jacobi Identity

$$\epsilon^{\mu\nu\alpha\beta} [A_\nu, [A_\alpha, A_\beta]]$$

$$= \frac{1}{3} \epsilon^{\mu\nu\alpha\beta} \{ (A_\nu, [A_\alpha, A_\beta]) + \text{cyclic perms.} \}$$

$$= 0.$$

$$\partial_\mu k^\mu = \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \text{Tr} \left[\partial_\mu A_\nu \partial_\alpha A_\beta + i\partial_\mu A_\nu [A_\alpha, A_\beta] + i\partial_\mu A_\nu \partial_\alpha [A_\beta, A_\gamma] \right]$$

$$= \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \text{Tr} \left[(\partial_\mu A_\nu + i[A_\mu, A_\nu]) (\partial_\alpha A_\beta + i[A_\alpha, A_\beta]) \right]$$

$$= \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \text{Tr} [G_{\mu\nu} G_{\alpha\beta}]$$

$$\frac{1}{2} k^\mu = \frac{1}{2} \text{Tr} [G_{\mu\nu} \tilde{G}^{\mu\nu}]$$

$$k^\mu = \epsilon^{\mu\nu\alpha\beta} \text{Tr} [A_\nu \partial_\alpha A_\beta + i\frac{2}{3} A_\nu A_\alpha A_\beta]$$

If θ -term is total derivative, why do we care? There are field configurations called instantons, for which $\int d^4x \frac{1}{2} \text{Tr} [G_{\mu\nu} \tilde{G}^{\mu\nu}] \neq 0$.

Our convention for $D_\mu = \partial_\mu + igA_\mu$
 $i\cancel{D} = i\cancel{D} - gA$

Interpreting the θ term,

Winding number

consider functions $g(\theta)$ w/ $g(\theta)$ is constrained to be complex number of modulus unity.



Mapping of $S_1 \rightarrow S_1$.

Mappings can be characterized by topological invariants (insensitive to continuous deformation)

$$V = \frac{i}{2\pi} \int_0^{2\pi} d\theta g \frac{dg^{-1}}{d\theta} \quad \leftarrow \text{winding number}$$

$$\delta g \Rightarrow i \delta \lambda g \quad \delta \lambda \text{ real}$$

$$\delta \left(g \frac{dg^{-1}}{d\theta} \right) = i \delta \lambda g \frac{dg^{-1}}{d\theta} + g \frac{d(-i \delta \lambda g^{-1})}{d\theta}$$

$$= i \delta \lambda g \left(\frac{dg^{-1}}{d\theta} \right) = i \frac{d}{d\theta} \delta \lambda g \frac{dg^{-1}}{d\theta} + g (-i \delta \lambda) \left(\frac{dg^{-1}}{d\theta} \right)$$

$$= -i \frac{d}{d\theta} \delta \lambda$$

(2)

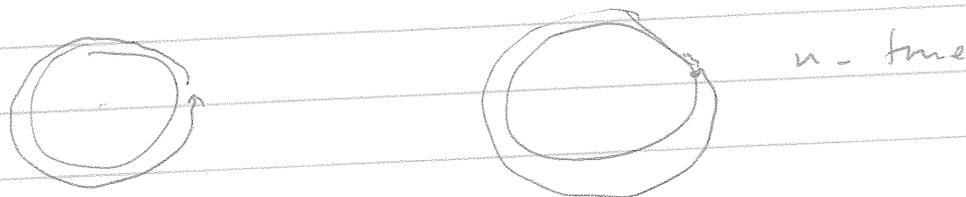
$$\delta v = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{d}{d\theta} \delta\lambda = \frac{1}{2\pi} (\delta\lambda(2\pi) - \delta\lambda(0)) = 0$$

(we are assuming λ & $\delta\lambda$ are single valued on θ .)

In this case easy to interpret v consider: $g(\theta) = e^{in\theta}$

$$v = \frac{-i}{2\pi} \int_0^{2\pi} d\theta g \frac{1}{d\theta} g^{-1} = \frac{n}{2\pi} \int_0^{2\pi} d\theta = n$$

v characterizes integer number of times $g(\theta)$ winds around unit circle $g_n(0)$.



relationship to gauge theory

~~$A_n \rightarrow 0$ at $x \rightarrow \infty$.~~

~~This means $A_n = g \partial_n g^{-1}$~~

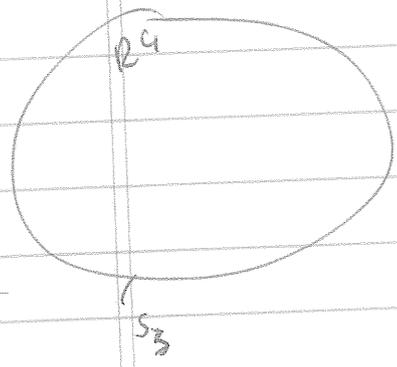
~~$A_n \rightarrow h A_n h^{-1} + h \partial_n h^{-1}$~~

~~$A_n \rightarrow \iff A_n = g \partial_n g^{-1}$~~

To see the relevance to gauge theory
 Consider solutions to Yang Mills equations
 in Euclidean space time. $(t, \vec{x}) \rightarrow (-it, \vec{x}) \equiv (x_4, \vec{x})$

To obtain a solution w/ finite action
 we must have $F_{uv} \rightarrow 0$ at $R \rightarrow \infty$ ($R = \sqrt{x_4^2 + \vec{x}^2}$)
 this means A_n must tend to pure
 gauge:

$$A_n(R \rightarrow \infty) = i g \partial_n g^{-1}$$



$$A_n \rightarrow U A_n U^{-1} + \frac{i}{g} U \partial_n U^{-1}$$

below
 $u \rightarrow s$
 $g \rightarrow 1$

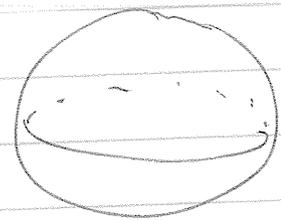
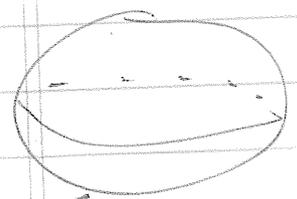
~~the loop at infinity~~

The surface at infinity is S^3 ($x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$)

now consider g for the case of $SU(2)$. Then
 an element of $SU(2)$ is

gk: $g = a + i \vec{b} \cdot \vec{\sigma}$ $g^{-1}g = 1 \rightarrow a^2 + b^2 = 1$
 $a^2 + b_1^2 + b_2^2 + b_3^2 = 1$

clearly the group space of $SU(2)$ is S^3



$S^3 = \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$

$g(\vec{\sigma}) = a(\vec{\sigma}) - i \vec{b}(\vec{\sigma}) \cdot \vec{\sigma}$

Sp(1)

$S_3 \rightarrow S_3$ maps also characterized by winding numbers

expression for winding number is

$$v = -\frac{1}{24\pi^2} \int d^3\theta \text{Tr} [\epsilon^{ijk} g \partial_i g^{-1} g \partial_j g^{-1} g \partial_k g^{-1}]$$

There are two steps to ~~showing~~ ^{demonstrating} correctness of this formulae.

1) showing that it is invariant under a local transformation. This means that v is a topological quantity.

$$\begin{aligned}
 g(\theta) &\rightarrow g(\theta) g(\theta') && \text{right multiply} \\
 &\approx g(\theta) (1 + i \delta\theta^a T^a) && g(\theta) \\
 \Rightarrow \delta g &= i g \delta\theta && \delta\theta = \delta\theta^a T^a
 \end{aligned}$$

$$\begin{aligned}
 \delta (g \partial_i g^{-1}) &= i \left[g \delta\theta^a \partial_i g^{-1} - g \partial_i (g \delta\theta^a) \right] \\
 &= i g \left[\partial_i \delta\theta^a \right] g^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \delta v &\approx \int d^3\theta \text{Tr} [\epsilon^{ijk} g \partial_i g^{-1} g \partial_j g^{-1} g \left[\partial_k \delta\theta^a \right] g^{-1}] \\
 &\quad \underbrace{\hspace{10em}}_{-g^i \partial_j g} \\
 &\quad \underbrace{\hspace{10em}}_{\text{cancel}}
 \end{aligned}$$

$$\approx - \int d^3\theta \text{Tr} [\epsilon^{ijk} \partial_i g^{-1} \partial_j g \partial_k \delta\theta^a] = 0 \quad (\text{integrate by parts \& antisymmetry})$$

We can show v is an integer

choose coord. so that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$
 $x^2 + x_4^2 = 1$

$$\int d^3x = \int d^4x \delta(1 - \sqrt{x^2 + x_4^2})$$

$$= 2 \int \frac{d^3x}{\sqrt{1-x^2}} = 8\pi \int_0^1 dr \frac{r^2}{\sqrt{1-r^2}} = 2\pi^2$$

Alt: $\int d^4x e^{-x_4^2 - x^2} = \left(\int_{-\infty}^{\infty} dx e^{-x^2} \right)^4 = \pi^2$

$$\int d^4x e^{-x_4^2 - x^2} = \int_0^{\infty} dr r^3 \int d\Omega_3 e^{-r^2}$$

$$= \int d\Omega_3 \frac{1}{2} \int du u e^{-u}$$

$$= \frac{1}{2} \int d\Omega_3 = \pi^2$$

Key point is

Standard mapping is $g = \frac{x_4 + i\vec{x} \cdot \vec{\sigma}}{r}$ $r = \sqrt{x_4^2 + x^2}$

$$g = \pm \sqrt{1 - \vec{v}^2} + i\vec{v} \cdot \vec{\sigma}$$

where $\vec{v} = \frac{\vec{x}}{r}$
 $0 < |\vec{v}| < 1$

near $\vec{v} = 1$

$$g \partial_i g^{-1} \Big|_{\vec{v}=0} = \sqrt{1 - \vec{v}^2} - i\vec{v} \cdot \vec{\sigma} \Big|_{\vec{v}=0} \frac{\partial}{\partial v_i} (\sqrt{1 - \vec{v}^2} - i\vec{v} \cdot \vec{\sigma}) = -i\sigma_i + \dots$$

$$\text{Tr} [\epsilon^{ijk} g \partial_i g^{-1} g \partial_j g^{-1} g \partial_k g^{-1}] \Big|_{\vec{v}=0} =$$

$$= (-i)^3 \text{Tr} [\epsilon^{ijk} \sigma_i \sigma_j \sigma_k] = (-i)^3 i \text{Tr} [\epsilon^{ijk} \partial_u \sigma_e]$$

$$= -12$$

key to evaluating this integral is to note that invariance under

$$g(\theta) \rightarrow g(\theta)g(\theta')$$

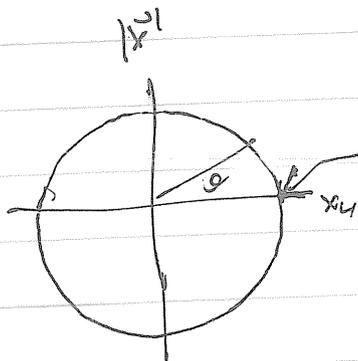
can be interpreted as as a coordinate transformation we can always rotate our coordinate system so that $\vec{x} = 0$ is any given point is $\vec{x} = 0$. so our integrand is constant everywhere on sphere.

$$V = \frac{-1}{24\pi^2} \int d^3x T_{ij} [e^{ijk} g_j g^{-1} g_j g^{-1} g_k s^{-1}]$$

$$= \frac{-1}{24\pi^2} \cdot -12$$

$$= \frac{1}{2} \quad \text{for } g = \frac{x_4 + i \vec{k} \cdot \vec{\sigma}}{r}$$

or



evaluate here for this mapping

$$\begin{aligned} \text{map } g &= \frac{x_4 + i \vec{k} \cdot \vec{\sigma}}{r} \\ &= \cos\theta + i \sin\theta \hat{n} \cdot \vec{\sigma} \end{aligned}$$

it is clear that result should change under $\theta \rightarrow \theta + \alpha$

other winding numbers

$$g = g_1, g_2$$

so $v = v_1 + v_2$

$$g_v = \left[\begin{array}{c} \cancel{\dots} \\ \frac{x_1 + i x_2}{r} \end{array} \right]^v$$

$v = \text{integers}$

topological argument determine $g_1 \rightarrow 1$ North pole
 $g_2 \rightarrow 1$ South pole

then integral is sum of g_1 integral, & g_2 integral

Next part is to show that all $g(\theta)$ can be deformed into one of the $g_v(\theta)$ standard

How is all this related to θ term?

~~$$S_\theta = \int d^4x \frac{1}{2} \text{Tr} [F_{\mu\nu} \wedge F_{\alpha\beta}]$$~~

$$S_\theta = \theta \int d^4x \frac{1}{16\pi^2} \text{Tr} [G_{\mu\nu} \tilde{G}_{\mu\nu}]$$

set $S=1$

$$2_n k^n = 2_n F^{\mu\nu\alpha\beta} \text{Tr} [A_\nu \wedge A_\alpha + i \frac{2}{3} A_\nu \wedge A_\alpha \wedge A_\beta]$$

~~$$= \frac{1}{2} \text{Tr} [G_{\mu\nu} \tilde{G}_{\mu\nu}]$$~~

Set 5

$$\int d^4x \text{Tr} [G_{\mu\nu} \tilde{G}^{\mu\nu}]$$

ℓℓ

$$E^{\mu\nu\alpha\beta} F_{\alpha\beta} = 2 \frac{\mu\nu\alpha\beta}{2} A_{\alpha} A_{\beta} \parallel$$

$$= 2 E^{\mu\nu\alpha\beta} (\partial_{\alpha} A_{\beta} - ig A_{\alpha} A_{\beta})$$

$$= g^2 \int d^4x \ 2 \partial^{\mu} k_{\mu}$$

$$= g^2 \int d^4x \ 2 \partial_{\mu} E^{\mu\nu\alpha\beta} \text{Tr} [A_{\nu} (\partial_{\alpha} A_{\beta} + ig \frac{2}{3} A_{\alpha} A_{\beta})]$$

$$= g^2 \int d^4x \ 2 \partial_{\mu} E^{\mu\nu\alpha\beta} \text{Tr} [A_{\nu} (\overset{\leftarrow 0 \text{ at int}}{\frac{F_{\alpha\beta}}{2}} - \frac{ig}{3} A_{\alpha} A_{\beta})]$$

$$= \frac{2}{3} \int d^4x \ 2 \partial_{\mu} E^{\mu\nu\alpha\beta} \cdot \text{Tr} [(ig A_{\nu})(ig A_{\alpha})(ig A_{\beta})]$$

$$= -\frac{2}{3} \int d^4x \ 2 \partial_{\mu} E^{\mu\nu\alpha\beta} \text{Tr} [(g \partial_{\nu} g^{-1})(g \partial_{\alpha} g^{-1})(g \partial_{\beta} g^{-1})]$$

$$= -\frac{2}{3} \int dS_3 \ E^{ijk} \text{Tr} [(g \partial_i g^{-1})(g \partial_j g^{-1})(g \partial_k g^{-1})] \quad i,j,k \perp \vec{r}$$

$$= + 16\pi^2 \checkmark$$

$$\Rightarrow \boxed{\frac{g^2}{16\pi^2} \int d^4x \ \text{Tr} [G_{\mu\nu} \tilde{G}^{\mu\nu}] = \checkmark}$$

Field configurations w/ non-vanishing ν are called instantons

$$\int d^4x (G_{\mu\nu} - \tilde{G}_{\mu\nu})(G_{\mu\nu} - \tilde{G}_{\mu\nu}) \geq 0$$

but $\tilde{G}_{\mu\nu} \hat{=} G_{\mu\nu} = G_{\mu\nu} G_{\mu\nu}$

$$\begin{aligned} \text{so } \frac{1}{2} \int d^4x \text{Tr}(G_{\mu\nu} G_{\mu\nu}) &\geq \frac{\pm 1}{2} \int d^4x \text{Tr}(G_{\mu\nu} \tilde{G}_{\mu\nu}) \\ &\geq \frac{8\pi^2 |\nu|}{5\pi} \end{aligned}$$

action is bounded from below. Bound

saturated by $G_{\mu\nu} = \pm \tilde{G}_{\mu\nu}$
 \uparrow linear diff. eq.

\pm instanton
 anti instanton

$$A_{\mu}^{\text{inst}} = \frac{r^2}{r^2 + \rho^2} g^{(1)} \partial_{\mu} [g^{(1)}]^{-1}$$

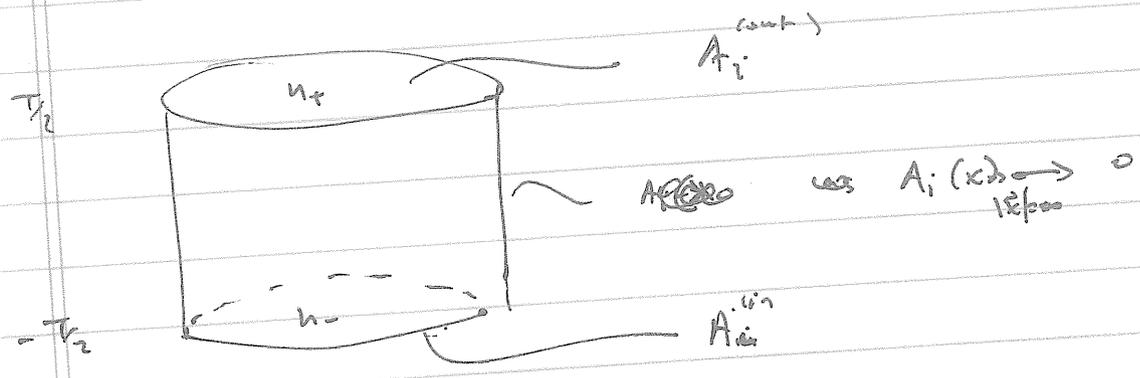
$$g^{(1)} = \frac{x_4 + i \vec{x} \cdot \vec{\sigma}}{r}$$

$$r = \sqrt{x_4^2 + \vec{x}^2}$$

$$A_{\mu}^{\text{inst}} = \begin{pmatrix} A_{\mu}^{\text{SU}(2) \text{ inst}} & \\ \hline & i \tau_3 \end{pmatrix}$$

SU(2) \rightarrow SU(N)

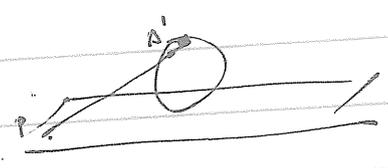
Interpreting the instanton solution
 consider $A_4 = 0$ gauge & then consider
 a vacuum \rightarrow vacuum transition amplitude



$A_i^{(in)}$ $A_i^{(out)}$ are pure gauge configurations.

$A_i^{(in)} = g \partial_i g^{-1}$ Furthermore $A_i(x) \Big|_{|x| \rightarrow \infty} = 0$

since $A_i(x \rightarrow \infty)$ are all the same we can
 compactify $\mathbb{R}^3 \rightarrow S^3$



$S^3: \mathbb{R}^3 + \text{"point at } \infty \text{"}$

$\Rightarrow n_{\pm} = \frac{-1}{24\pi^2} \int d^3x \text{Tr} \left[\epsilon^{ijk} g \partial_j g^{-1} g \partial_k g^{-1} g \partial_i g^{-1} \right]$

is a winding number associated w/ ~~vacua~~ ^(out) _(in) vacua
 and ~~we~~ we identify

$V = n_+ - n_-$

as change in winding number.

(instanton is interpreted as a tunneling event from
 a vacuum w/ n_- winding number to a vacuum w/ n_+ winding number)

~~vacuum~~
vacuum of gauge theories

$|n\rangle$ $n =$ winding number

$$U_{(n)} |n\rangle = |n+1\rangle$$

$U_{(n)}$ is a large gauge transformation
 \uparrow
line

$$\frac{A_m=0}{n=0} \quad A_m = \frac{i}{g} \sum_{n=1} g_n \partial_n S_i^{-1}$$

Both pure gauge but not physically equivalent

gauge invariant vacuum (up to a phase)

$$|0\rangle = \sum_n e^{in\theta} |n\rangle$$

$$U_{(n)} |0\rangle = e^{in\theta} |0\rangle$$

$$\langle 0 | e^{-iHt} |0\rangle \propto \delta(\theta - \theta') \quad \text{since } U_{(n)} H U_{(n)}^{-1} = H$$

~~$$\langle 0 | e^{-iHt} |0\rangle = \sum_{m,n} e^{i\theta(m-n)} \langle m | e^{iHt} |n\rangle$$~~

so in path integral $S = S_{ocp} + \theta V = S_{ocp} + \frac{\theta g^2}{16\pi^2} \int d^4x Tr F_{\mu\nu}^2$

$$n_+ = n_- + \nu$$

(11)

$$\langle \sigma' | e^{-iHt} | \sigma \rangle = \sum_{n_+, n_-} e^{+in_+ \sigma'} e^{-in_- \sigma} \langle n_+ | e^{-iHt} | n_- \rangle$$

$$= \sum_{n_-} e^{+in_-(\sigma-\sigma')} \sum_{\nu} e^{i\nu\sigma'} \langle n_+ | e^{-iHt} | n_- \rangle$$

$$\propto (\sigma-\sigma') \sum_{\nu} e^{i\nu\sigma'} \langle n_+ | e^{-iHt} | n_- \rangle$$

Correct
~~Correct~~

Correct path integral expression for transition amplitude is

$$\sum_{\nu} \int [dA_m]_{\nu} e^{iS(A_m) + \theta \nu}$$

$$= \sum_{\nu} \int [dA_m]_{\nu} e^{iS(A_m) + \theta \frac{g^2}{16\pi^2} \int d^4x \text{Tr}(G_{\mu\nu} \tilde{G}^{\mu\nu})}$$

(2)

Next, we wish to understand how anomaly resolves $\pi^0 \rightarrow \gamma\gamma$ ~~puzzle~~ puzzle. Historically, resolving this puzzle is what led to discovery of anomaly.

To understand ~~the~~ $\pi^0 \rightarrow \gamma\gamma$ puzzle we need to make a long digression and 1st understand how we study π 's in QCD. The essential tool is EFT - chiral ~~per~~ perturbation theory.

We know $\langle 0 | \bar{q}_L q_R - \bar{q}_R q_L | 0 \rangle \neq 0$ breaks

$$SU_L(2) \times SU_R(2) \rightarrow SU(2)_{LR}$$

(we consider 2-flavor QCD in what follows) and this implies existence of 3 Goldstone bosons (π^+, π^0). We now want to know how to write down a theory to describe the physics of these particles as well as their interactions. Key concepts

- identifying degrees of freedom
- incorporating ~~sym~~ symmetries
- power counting.

for 2 flavors $SU(2) \times SU(2) \sim O(4)$
 so the pattern of symmetry breaking is similar
 to what we saw in $O(N)$ models earlier

e.g. $O(4) = \mathcal{L} = \frac{1}{2} \partial_\mu \vec{\phi} \partial^\mu \vec{\phi} = \frac{\lambda}{4} (\vec{\phi}^2 - v^2)^2$ $\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$

Earlier we saw a convenient parametrization

$\phi = e^{i \vec{T} \cdot \frac{\vec{\pi}}{v}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ v + \rho \end{pmatrix}$ $\langle \phi \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix}$ $\Sigma \equiv e^{i \vec{T} \cdot \frac{\vec{\pi}}{v}}$

the T^a here are broken generators $T^a \langle \phi \rangle = T^a \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix} \neq 0$

$\mathcal{L} = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \lambda v^2 \rho^2 + \lambda v \rho^3 + \frac{\lambda}{4} \rho^4$ ← massive scalar

~~$\frac{1}{2} \partial_\mu \vec{\pi} \partial^\mu \vec{\pi} + \frac{(\vec{\pi} \cdot \hat{n})^2}{2} \hat{n}^T \partial_\mu \Sigma^T \partial_\mu \Sigma \hat{n}$~~ $\frac{(\nu + \rho)^2}{2} \hat{n}^T \partial_\mu \Sigma^T \partial_\mu \Sigma \hat{n}$
 ↑ ↑ goldstone

typically we are interested in low-energy limit
 integrate out ρ . ($m_\rho = \sqrt{2\lambda} v$)

$\mathcal{L} = \frac{v^2}{2} \hat{n}^T \partial_\mu \Sigma^T \partial_\mu \Sigma \hat{n}$

But since $O(4) \sim SU(2) \times SU(2)$ we ought
 to write this in a way which looks
 more like $SU(2) \times SU(2)$ chiral theory.

Basic elements of chiral pert theory

$\Sigma = e^{2i\vec{\pi} \cdot \vec{T} / f}$ $\vec{T} = \frac{\vec{T}}{2}$ are conventionally normalized generators.

transformation law under $SU_L(2) \times SU_R(2)$ $\Sigma \rightarrow L \Sigma R^\dagger$

1) unbroken subgroup $SU_{V(2)}$ isospin $L=R=e^{-i\vec{\theta} \cdot \vec{T}}$

$\Sigma \rightarrow e^{-i\vec{\theta} \cdot \vec{T}} e^{2i\vec{\pi} \cdot \vec{T} / f} e^{i\vec{\theta} \cdot \vec{T}} \Rightarrow \vec{T} \cdot \vec{\pi} \rightarrow e^{-i\vec{\theta} \cdot \vec{T}} \vec{\pi} \cdot \vec{T} e^{-i\vec{\theta} \cdot \vec{T}}$

π 's transform linearly under unbroken group

2) ~~axial~~ axial transformation - broken generators $L=R^\dagger=e^{-i\vec{\theta} \cdot \vec{T}}$

$\Sigma \rightarrow e^{-i\vec{\theta} \cdot \vec{T}} e^{2i\vec{\pi} \cdot \vec{T} / f} e^{-i\vec{\theta} \cdot \vec{T}}$

$\vec{\pi} \rightarrow \vec{\pi} - f \vec{\theta}$ non-linear transformation.

3) $\Sigma \Sigma^\dagger = \mathbb{1}$ must have derivative interactions

$\text{Tr}[\partial_\mu \Sigma \partial^\mu \Sigma^\dagger] + \text{Tr}[\partial_\mu \Sigma \partial_\nu \Sigma^\dagger \partial^\nu \Sigma \partial^\mu \Sigma^\dagger] + \dots$

derivatively coupled

\Rightarrow Field theory of Σ incorporates all relevant constraints of $SU_L(2) \times SU_R(2) \rightarrow SU_{V(2)}$ on dynamics of pions.

How do we incorporate explicit symmetry breaking?

$$\mathcal{L}_{\text{QCD}} = -m_q (\bar{q}_L q_R + \bar{q}_R q_L)$$

imagine fictitious field $\overset{\text{spurion}}{M}$ which transforms as
 $M \rightarrow LM R^\dagger$

and acquires a vev $\langle M \rangle = m_q$. (can replace

$$\mathcal{L}_{\text{QCD}} = -\bar{q}_L M q_R - \bar{q}_R M^\dagger q_L$$

now we write down most general EFT w/ Σ, M
 then $M \rightarrow \langle M \rangle = m_q$. Explicit symmetry breaking
 is transmitted to effective theory:

lowest order ch. p.T. lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{f^2}{8} \text{Tr} [\partial_\mu \Sigma \partial^\mu \Sigma^\dagger] - \frac{B f^2}{8} \text{Tr} [M \Sigma + M^\dagger \Sigma^\dagger] \\ &= \frac{f^2}{8} \text{Tr} [\partial_\mu \Sigma \partial^\mu \Sigma^\dagger] - \frac{B f^2}{8} \text{Tr} [m_q (\Sigma + \Sigma^\dagger)^\dagger] \end{aligned}$$

Interpretation parameter B. (Gell-Mann-Lakes-Kenner)

$$Z_{acc} = \int D\mu D\bar{\nu} D\lambda e^{i \int d^4x \bar{q}_a (i \not{\partial} - m_{q_a}) q_a}$$

$$\frac{1}{Z_{acc}} \left. \frac{\delta Z_{acc}}{\delta (\mu_a)_{ab}} \right|_{(\lambda_a)_{ab} = 0} = \langle 0 | \bar{q}_a q_b | 0 \rangle \cdot VT$$

$$= \int d^4x \langle 0 | \bar{q}_a q_b(x) | 0 \rangle$$

$$\int d^4x = VT$$

$$Z_{exp} = \int D\Sigma \cdot e^{i \int d^4x \left[\frac{f^2}{8} \text{Tr} [\Sigma^\dagger \partial_\mu \Sigma] - \frac{c}{2} \text{Tr} [m(\Sigma + \Sigma^\dagger)] \right]}$$

$$\frac{1}{Z_{exp}} \left. \frac{\delta Z_{exp}}{\delta (\mu_a)_{ab}} \right|_{(\lambda_a)_{ab} = 0} = \frac{c}{2} \langle 0 | (\Sigma + \Sigma^\dagger)_{ab} | 0 \rangle \int d^4x$$

$$= c \text{Sub} \int d^4x$$

$$c = \frac{B f^2}{2}$$

$$\Rightarrow c = \frac{B f^2}{2} = \langle 0 | \bar{q} q | 0 \rangle$$

$$\langle 0 | \bar{q}_a q_b | 0 \rangle = \text{Sub} \langle 0 | \bar{q} q | 0 \rangle$$

$$\langle 0 | \bar{q} q | 0 \rangle = \langle 0 | \bar{u} u | 0 \rangle = \langle 0 | \bar{d} d | 0 \rangle = \langle 0 | \bar{s} s | 0 \rangle$$

For $SU(2)$ $m_u = m_d = m_s$

$$- \frac{c}{2} \text{Tr} [m(\Sigma + \Sigma^\dagger)]$$

$$= - \frac{c}{2} \text{Tr} [m \left(1 - \frac{4}{f^2} \pi^2 \right)]$$

$$\bar{q} = \begin{pmatrix} \bar{u}_0/\sqrt{2} & \bar{s}^\dagger \\ \bar{d}^- & -\bar{u}_0/\sqrt{2} \end{pmatrix}$$

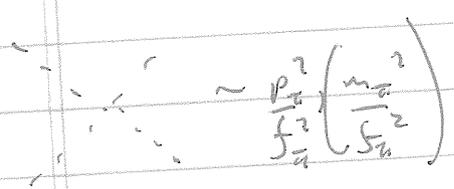
$$\text{Tr} [\pi^2] = \pi_0^2 + 2 \bar{q}^\dagger \bar{q}$$

$$= + \frac{2c}{f^2} m_q \text{Tr} [\pi^2] = \frac{2c}{f^2} m_q (\pi_0^2 + 2 \bar{q}^\dagger \bar{q}) = - \frac{1}{2} m_q^2 (\pi_0^2 + 2 \bar{q}^\dagger \bar{q})$$

$$\Rightarrow m_\pi^2 = - \frac{4c}{f^2} m_q = \text{Sub} \left[\frac{2 \langle 0 | \bar{q} q | 0 \rangle (m_u + m_d)}{f^2} \right] = m_\pi^2$$

Power Counting for $\pi\pi$ scattering

$$\mathcal{L}_0 = \frac{f^2}{8} \text{Tr} [\partial_\mu \Sigma \partial^\mu \Sigma^\dagger] - \frac{\beta f^2}{4} \text{Tr} [m_q (\Sigma + \Sigma^\dagger)]$$



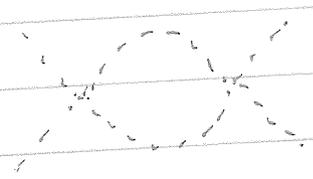
$p^2 \rightarrow s, t, u$

$$s = (p_1 + p_2)^2$$

$$t = (p_1 - p_3)^2$$

$$u = (p_1 - p_4)^2$$

Loops



$$\sim \frac{P_{\pi\pi}^4}{(4\pi)^2 f_\pi^4} \ln \left(\frac{P_{\pi\pi}^4}{m_\pi^2} \right) + \frac{1}{\epsilon}$$

$(4\pi)^2 \leftarrow$ typical factor obtained from one-loop diagram.

Note that $\frac{1}{\epsilon} P_{\pi\pi}^4$ cannot be cancelled

$$\left(\text{or } \frac{1}{\epsilon} P_{\pi\pi}^2 m_\pi^2, \frac{1}{\epsilon} m_\pi^4 \right)$$

by counterterm in LO Lagrangian $(\sim P_{\pi\pi}^2, m_\pi^2)$

This is Ok since at NLO we expect new contributions to EFT:

~~$$\mathcal{L} = \frac{f^2}{8} \text{Tr} [\partial_\mu \Sigma \partial^\mu \Sigma^\dagger]^2 + a_2 \text{Tr} [\partial_\mu \Sigma \partial_\nu \Sigma^\dagger] \text{Tr} [\partial^\mu \Sigma \partial^\nu \Sigma^\dagger]$$

$$+ a_3 \text{Tr} [\partial_\mu \Sigma \partial^\mu \Sigma^\dagger \partial_\nu \Sigma \partial^\nu \Sigma^\dagger] + a_4 \text{Tr} [\partial_\mu \Sigma \partial^\mu \Sigma^\dagger] \text{Tr} [\beta_0 m_q (\Sigma + \Sigma^\dagger)]$$

$$+ a_5 \text{Tr} [\partial_\mu \Sigma]$$~~

$$P_{\pi\pi}^4 \quad L_4 \sim P_{\pi\pi}^4 m_\pi^2$$

subleading logarithms (for SU(3) x PT).

21

$$\mathcal{L} = \alpha_1 \text{Tr} [\partial_\mu \Sigma^\dagger \partial^\mu \Sigma]^2 + \alpha_2 \text{Tr} [\partial_\mu \Sigma \partial_\nu \Sigma^\dagger] \text{Tr} [\partial_\mu \Sigma \partial_\nu \Sigma^\dagger] + \alpha_3 \text{Tr} [\partial_\mu \Sigma \partial^\mu \Sigma^\dagger \partial_\nu \Sigma \partial^\nu \Sigma^\dagger] + \alpha_4 \text{Tr} [\partial_\mu \Sigma \partial^\mu \Sigma^\dagger] \text{Tr} [B_0 u_\mu (\Sigma + \Sigma^\dagger)] + \alpha_5 \text{Tr} [\partial_\mu \Sigma \partial^\mu \Sigma^\dagger u_\mu B (\Sigma + \Sigma^\dagger)] + \alpha_6 \text{Tr} [u_\mu B (\Sigma + \Sigma^\dagger)]^2 + \alpha_7 \text{Tr} [u_\mu B (\Sigma + \Sigma^\dagger)]^2 + \alpha_8 \text{Tr} [u_\mu B \Sigma^\dagger u_\nu \Sigma + \xi u_\mu \Sigma u_\nu]$$

$O(p^4)$

$O(m_\pi^2 p^2)$

$O(m_\pi^4)$

Then including these terms: ($m_\pi \rightarrow 0$)



$$= \frac{P_n^4}{(4\pi f_n^2) f_n^2} \ln \left(\frac{P_n^2}{m^2} \right) \sim f_n^4 \alpha_i(m)$$

m -dependence between $\ln(m)$ & $\alpha_i(m)$ ~~cancel~~ ~~cancel~~ ^{cancel}
 For natural choice of m ($\sim 1.5 \text{ GeV}$) we expect these
 are the same order. Then expansion parameter

is

$$\frac{P_n^2 (m_\pi^2)}{(4\pi f_n^2)^2} \sim \frac{P_n^2 (m_\pi^2)}{\Lambda_\chi^2}$$

$$\Lambda_\chi = 4\pi f_n \sim 1.5 \text{ GeV}$$

~~Let $\alpha = \frac{\vec{p} \cdot \vec{e}}{v}$~~

Let $\alpha = \frac{\vec{p} \cdot \vec{e}}{v}$

$$\partial^n \Sigma = \partial^n e^{i\alpha} = \partial^n \left(1 + i\alpha + \frac{\alpha^2}{2} + i\frac{\alpha^3}{3!} \right)$$

$$= i\partial_n \alpha - \frac{\partial_n \alpha^2}{2} + i\frac{\partial_n \alpha^3}{3!}$$

$$\text{Tr}[\partial^n \Sigma \partial_m \Sigma^\dagger] = \text{Tr} \left[\left(i\partial_n \alpha - \frac{\partial_n \alpha^2}{2} + i\frac{\partial_n \alpha^3}{3!} \right) \left(-i\partial_m \alpha - \frac{\partial_m \alpha^2}{2} + i\frac{\partial_m \alpha^3}{3!} \right) \right]$$

$$= \text{Tr}[\partial_n \alpha \partial_m \alpha] + \text{Tr} \left[\partial_n \alpha \frac{\partial_m \alpha^3}{3!} + \frac{\partial_n \alpha^3}{3!} \partial_m \alpha \right] + \frac{1}{4}$$

$$= \text{Tr}[\partial_n \alpha \partial_m \alpha] - \text{Tr} \left[\partial_n \alpha \frac{\partial_m \alpha^3}{3!} + \frac{\partial_n \alpha^3}{3!} \partial_m \alpha \right] + \frac{1}{4} \text{Tr}[\partial_n \alpha^2 \partial_m \alpha^2]$$

$$= \text{Tr}[\partial_n \alpha \partial_m \alpha] - \frac{1}{3} \text{Tr}[\partial_n \alpha \partial_m \alpha^3] + \frac{1}{4} \text{Tr}[\partial_n \alpha^2 \partial_m \alpha^2]$$

$$= \text{Tr}[\partial_n \alpha \partial_m \alpha] - \frac{2}{3} \text{Tr}[\partial_n \alpha \partial_m \alpha^2] - \frac{1}{3} \text{Tr}[\partial_n \alpha^2 \partial_m \alpha]$$

$$+ \frac{1}{2} \text{Tr}[\partial_n \alpha \partial_m \alpha^2] + \frac{1}{2} \text{Tr}[\partial_n \alpha^2 \partial_m \alpha]$$

$$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} \quad -\frac{2}{3} + \frac{1}{2} = -\frac{1}{6}$$

$$= \text{Tr}[\partial_n \alpha \partial_m \alpha] + \frac{1}{6} \text{Tr}[\partial_n \alpha \alpha \partial_m \alpha \alpha - \partial_n \alpha \partial_m \alpha \alpha \alpha]$$

$$+ \frac{1}{6} \text{Tr}[\partial_n \alpha [\alpha, \partial_m \alpha] \alpha]$$

$$= \text{Tr}[\partial_n \alpha \partial_m \alpha] + \frac{1}{12} \text{Tr}[\alpha, \partial_m \alpha] [\alpha, \partial_n \alpha]$$

$$\text{Tr}[\alpha, \gamma] [\alpha, \gamma]$$

~~$$\text{Tr}[\alpha, \gamma]$$~~

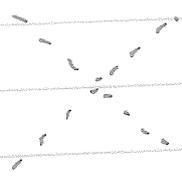
$$\text{Tr}[(xy - yx)(xy - yx)]$$

$$2 \text{Tr}[xyxy - yxyx]$$

$$2 \text{Tr}[xyxy - yxyx]$$

$$2 \text{Tr}[y^2 x^2]$$

$$\mathcal{L} = \frac{1}{4} \text{Tr}[\partial_n \bar{\psi} \partial_m \psi] + \frac{1}{12v^2} \text{Tr}[\bar{\psi}, \partial_n \bar{\psi}] [\psi, \partial_m \psi] + \dots$$

key point  $\sim \frac{P^i P^m}{v^2}$

Derivatively coupled - weakly interacting at low energies.

In the linear σ model, $v = \langle \phi \rangle$.

In QCD, there is no scalar field ϕ . instead a composite field $\langle \bar{q}q \rangle \neq 0$ (v)

$v = f_\pi$ - pion decay constant $\langle \bar{u}^a | j_5^b | 0 \rangle = -i f_\pi p^b$

(we will work this out later.)

Also need to understand relationship between $\langle \bar{q}q \rangle$ and parameters in chiral lagrangian.

But first work out some general features of theories w/ SSB chiral symmetry.

$\bar{\psi} = e^{i\alpha^a T^a} \langle \psi_0 \rangle$ $\langle 0 | \bar{q}_a q_a | 0 \rangle \rightarrow \langle 0 | \bar{q}_a e^{i\alpha^b T^b} q_a | 0 \rangle$ unit generate

regardless of nature of order parameter, G.B. can be ~~regenerated~~ obtained by acting on vacuum w/ broken generators and promoting parameters to space-time fields.
 sym. has generated by

$$\psi = \begin{pmatrix} \frac{u'}{\sqrt{2}} \\ \frac{v'}{\sqrt{2}} \\ \frac{v'}{\sqrt{2}} \\ \sigma \end{pmatrix} = \begin{pmatrix} u \\ v \\ v \\ \sigma \end{pmatrix} \quad \Sigma =$$

let $\Sigma = \sigma \mathbb{1} + i \vec{c} \cdot \vec{n}$ $\frac{c}{c}$ are pauli matrices.
 $\text{Tr}[\tau_i \tau_j] = 2\delta_{ij}$

$$\text{Tr}[\Sigma \Sigma^\dagger] = \text{Tr}[(\sigma \mathbb{1} + i \vec{c} \cdot \vec{n})(-\sigma \mathbb{1} + i \vec{c} \cdot \vec{n})]$$

$$= 2(\sigma^2 + \vec{n}^2)$$

$$\text{Tr}[\partial_\mu \Sigma \partial^\mu \Sigma^\dagger] = 2(\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \vec{n} \cdot \partial^\mu \vec{n})$$

$$\mathcal{L} = \frac{1}{4} \text{Tr}[\partial_\mu \Sigma \partial^\mu \Sigma^\dagger] - \frac{\lambda}{4} \left(\frac{1}{2} \text{Tr}[\Sigma^\dagger \Sigma] - v^2 \right)^2$$

symmetry transformation in terms of Σ is

$$\Sigma \rightarrow L \Sigma R^\dagger \quad L \in \text{SU}_L(2) \quad R \in \text{SU}_R(2)$$

$$\langle \Sigma \rangle = \begin{pmatrix} v \\ \vec{n} \end{pmatrix} \quad \text{under } \text{SU}_L(2) \times \text{SU}_R(2)$$

$$\rightarrow L \langle \Sigma \rangle R^\dagger = v L R^\dagger \quad \text{invariant under}$$

very left invariant in $L=R$ i.e. $\text{SU}_L(2) \times \text{SU}_R(2) \rightarrow \text{SU}_{\text{diag}}(2)$

in the limit $m_\sigma \rightarrow \infty$ (massive states decouple)

$$\Sigma \approx (v + \sigma) e^{i \vec{n} \cdot \vec{\tau} / v} \quad \sigma \rightarrow 0$$

$$\mathcal{L} = \frac{v^2}{4} \text{Tr}[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma]$$

$$= \frac{v^2}{4} \partial_\mu \vec{n} \cdot \partial^\mu \vec{n}$$

Computing axial currents in XPT (1.0).

$$\mathcal{L} = \frac{f^2}{8} \text{Tr} [\partial_\mu \Sigma \partial^\mu \Sigma^\dagger]$$

$$SU_L(3) : \Sigma \rightarrow L \Sigma = (1 + i\alpha(x)) \Sigma \quad \Sigma^\dagger \rightarrow \Sigma^\dagger (1 - i\alpha(x))$$

$$\delta \mathcal{L} = \frac{f^2}{8} \text{Tr} [\partial_\mu (i\alpha \Sigma) \partial^\mu \Sigma^\dagger - \partial_\mu \Sigma (\partial_\mu \Sigma^\dagger i\alpha)]$$

$$= \frac{f^2}{8} \text{Tr} [i\partial_\mu \alpha(x) (\Sigma \partial^\mu \Sigma^\dagger - \partial_\mu \Sigma \Sigma^\dagger)]$$

$$= \frac{f^2}{4} \text{Tr} [i\partial_\mu \alpha(x) \Sigma \partial^\mu \Sigma^\dagger]$$

$$j_{L\mu} = -i \frac{f^2}{4} \Sigma \partial_\mu \Sigma^\dagger$$

likewise for $SU_R(3) : \Sigma \rightarrow \Sigma R^\dagger = \Sigma (1 - i\alpha'(x))$

similar set of ~~manipulations~~ manipulations leads

$$j_{R\mu} = i \frac{f^2}{4} \Sigma^\dagger \partial_\mu \Sigma$$

$$\Rightarrow j_{A\mu} = j_{L\mu} - j_{R\mu} = \frac{if^2}{4} \Sigma \partial_\mu \Sigma^\dagger - \Sigma^\dagger \partial_\mu \Sigma$$

to lowest order $= i \frac{f^2}{4} \left(\partial_\mu \left(\frac{-2i\pi}{f} \right) - \partial_\mu \frac{2i\pi}{f} \right) = f \partial_\mu \pi$

②

Higher order terms $\sim \frac{1}{f} [\pi[\pi, \partial_n \pi]]$

$$\begin{aligned} \langle 0 | \int_{\mu\nu} | \bar{\pi}(p) \rangle &= \langle 0 | \int \partial_n \bar{u} + \dots | \pi \rangle \\ &= -i f P_n \end{aligned}$$

$$\Rightarrow \boxed{f = f_\pi} \quad (\text{to lowest order in } \chi PT)$$

f_π - pion decay constant

In the σ -model $\langle \sigma \rangle = f_\pi$ no such
interpretation in QCD.

Impact of anomalies of ChPT

our Lagrangian has a problem

$$\partial_n j_L^M = \partial_n j_R^M = 0$$

not consistent w/ anomaly

showing current conservation:

~~showing~~

$$\int d^4x \partial_\mu j^\mu(x)$$

$$= \int d^4x \partial_\mu (\bar{\psi} \gamma^\mu \psi)$$

$$0 = \partial_\mu j^\mu$$

$$\partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0$$

$$\partial_\mu \bar{\psi} \gamma^\mu \psi + \bar{\psi} \gamma^\mu \partial_\mu \psi = 0$$

$$\begin{aligned} \partial_\mu \bar{\psi} \gamma^\mu \psi &= -\bar{\psi} \gamma^\mu \partial_\mu \psi \\ &= -\partial_\mu \bar{\psi} \gamma^\mu \psi \end{aligned}$$

$$\partial_\mu \bar{\psi} \gamma^\mu \psi = 0$$

$$\partial_\mu \bar{\psi} \gamma^\mu \psi = 0$$

$$\int d^4x \partial_\mu (\bar{\psi} \gamma^\mu \psi)$$

$$= \int d^4x \partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0$$

$$= \int d^4x \partial_\mu (\bar{\psi} \gamma^\mu \psi)$$

$$= \int d^4x \partial_\mu (\bar{\psi} \gamma^\mu \psi)$$

$$\int d^4x \partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0$$

$$\Rightarrow \int d^4x \partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0$$

like wise $\partial_\mu \bar{\psi} \gamma^\mu \psi = 0$

$$\int d^4x \partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0$$

$$= \int d^4x \partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0$$

$$\boxed{\partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0}$$

(conservation of ~~total~~ L & R currents)

$$\sigma^2 = -\sigma^2$$

e.o.m

$$\begin{aligned}
\delta I &\leftarrow T_n \left| \frac{\partial \mathcal{L}}{\partial \dot{z}^m} \right|_{n1} \rightarrow \left[\left(+\frac{\partial \mathcal{L}}{\partial \dot{z}^m} \right) \dot{z}^m - \mathcal{L} \right]_{n1} \\
&= T_n \left| \frac{\partial \mathcal{L}}{\partial \dot{z}^m} \dot{z}^m - \mathcal{L} \right|_{n1} \\
&\quad - \left(\frac{\partial \mathcal{L}}{\partial \dot{z}^m} \dot{z}^m - \mathcal{L} \right)_{n1} \\
&= T_n \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{z}^m} \dot{z}^m + \mathcal{L} \right)_{n1} - \left(\frac{\partial \mathcal{L}}{\partial \dot{z}^m} \dot{z}^m - \mathcal{L} \right)_{n1} \right] \\
&= 2 T_n \left[\frac{\partial \mathcal{L}}{\partial \dot{z}^m} \dot{z}^m \right]_{n1} \\
&= 2 T_n \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{z}^m} \dot{z}^m + \mathcal{L} \right)_{n1} - \mathcal{L} \right]_{n1} \\
&= 2 T_n \left[\frac{\partial \mathcal{L}}{\partial \dot{z}^m} \dot{z}^m \right]_{n1} \\
&\Rightarrow \underline{\underline{\frac{\partial \mathcal{L}}{\partial \dot{z}^m} \dot{z}^m = 0}}
\end{aligned}$$

Let's go back to ChPT & effect of the anomalies. Leading order

$$\mathcal{L}_\pi = \frac{f^2}{8} \text{Tr} [\partial_\mu \Sigma^\dagger \partial^\mu \Sigma]$$

$$j_\mu^M = -i \frac{f^2}{4} \Sigma \partial^\mu \Sigma^\dagger$$

$$j_R^M = -i \frac{f^2}{4} \Sigma^\dagger \partial^\mu \Sigma$$

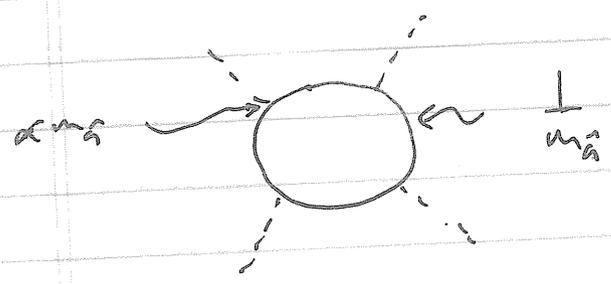
$$\partial_\mu j_\mu^M = \partial_\mu j_R^M = 0$$

To incorporate anomalies, add helicity quart \hat{g}

$$\mathcal{L} = \bar{q}_L i \not{\partial} q_L + \bar{q}_R i \not{\partial} q_R - m_q (\bar{q}_L \hat{g} q_R + \bar{q}_R \hat{g} q_L) + \mathcal{L}_\pi$$

This theory will reproduce anomalies because of \hat{g} (which gives the same anomalies as QCD.) We would like to take $m_q \rightarrow \infty$ and we will recover QCD (anomalies are m_q independent so we will still ~~recover~~ keep anomaly effects.)

Can't integrate out \hat{g} because of non-derivative couplings of G.B.



we can get rid of the non-derivative couplings by making field redefinition:

$$(i) \quad \begin{aligned} q_L &\rightarrow \Sigma(s) q_L & \Sigma(s=0) &= 1 \\ q_R &\rightarrow q_R & \Sigma(s=1) &= \Sigma \end{aligned}$$

$$S_{\text{opt}} = \int_x \bar{\hat{q}}_L (i\not{\partial} - A_L(s)) \hat{q}_L + \bar{\hat{q}}_R i\not{\partial} \hat{q}_R - \int_x m_{\hat{q}} \left(\bar{\hat{q}}_L \Sigma^\dagger(s) \Sigma \hat{q}_R + \bar{\hat{q}}_R \Sigma^\dagger(s) \Sigma \hat{q}_L \right) + \Delta(s)$$

~~background gauge field~~ $A_L^M = -i \Sigma^\dagger \partial^M \Sigma$

$\Delta(s)$ is generated because the transformation (1) is chiral and for $s \neq 1$ there is a (background) gauge field A_L^M which will lead to non-conservation of J_L^M .

for $s=1$:
$$\int_x \mathcal{L} = \int_x \bar{\hat{q}}_L (i\not{\partial} + A_L(1)) \hat{q}_L + \bar{\hat{q}}_R i\not{\partial} \hat{q}_R - m_{\hat{q}} \bar{\hat{q}}_L \hat{q}_R + \Delta(1)$$

now there are no non-derivative couplings of G.B.'s to \hat{q} so we can take $m_{\hat{q}} \rightarrow \infty$

$$S_{\text{opt}} = \int_x \mathcal{L}_x + \Delta(1)$$

④

computing change in action going from $S \rightarrow S + \delta S$

$$\delta q_L = q_L(s + \delta s) - q_L(s) = [\Sigma(s + \delta s) - \Sigma(s)] q_L$$

$$= \delta s \Sigma^\dagger \partial_s \Sigma q_L$$

might think $\Sigma(s + \delta s) - \Sigma(s) = \delta s \partial_s \Sigma$

but if $\delta q_L = \epsilon q_L$ $\delta q_L^\dagger = -\epsilon^\dagger q_L^\dagger$ since $\delta q_L q_L = 0$
 for a unitary transformation. $\Rightarrow \underline{\epsilon^\dagger = -\epsilon}$

can check $(\epsilon^\dagger \partial_s \epsilon)^\dagger = \partial_s \epsilon^\dagger \epsilon = -(\epsilon^\dagger \partial_s \epsilon) \checkmark$

$$\delta S \Delta(s) = \int d^4x \left[\dots \right]$$

$$= -i \delta s \text{Tr} \left[\Sigma^\dagger \partial_s \Sigma \partial_\mu j_\mu^h \right]$$

$$A_\mu^L = -i \epsilon^\dagger \partial_\mu \epsilon$$

$$\partial_\mu j_\mu^h = \frac{1}{16\pi^2} \partial_\mu k^\mu \quad k_\mu = \epsilon^{\mu\nu\alpha\beta} \left(A_\nu^L \partial_\alpha A_\beta^L + i \frac{2}{3} A_\nu^L A_\alpha^L A_\beta^L \right)$$

$$\partial_\mu k^\mu = \epsilon^{\mu\nu\alpha\beta} \partial_\mu \left[A_\nu^L \partial_\alpha A_\beta^L + i \frac{2}{3} A_\nu^L A_\alpha^L A_\beta^L \right]$$

$$= \epsilon^{\mu\nu\alpha\beta} \partial_\mu \left[(\epsilon^\dagger \partial_\nu \epsilon) \partial_\alpha (\epsilon^\dagger \partial_\beta \epsilon) + \frac{2}{3} (\epsilon^\dagger \partial_\nu \epsilon) (\epsilon^\dagger \partial_\alpha \epsilon) (\epsilon^\dagger \partial_\beta \epsilon) \right]$$

$$\epsilon^\dagger \partial_\alpha \epsilon \epsilon^\dagger = -\partial_\alpha \epsilon^\dagger$$

$$= -\frac{1}{3} \epsilon^{\mu\nu\alpha\beta} \partial_\mu \epsilon^\dagger \partial_\nu \epsilon \partial_\alpha \epsilon^\dagger \partial_\beta \epsilon$$

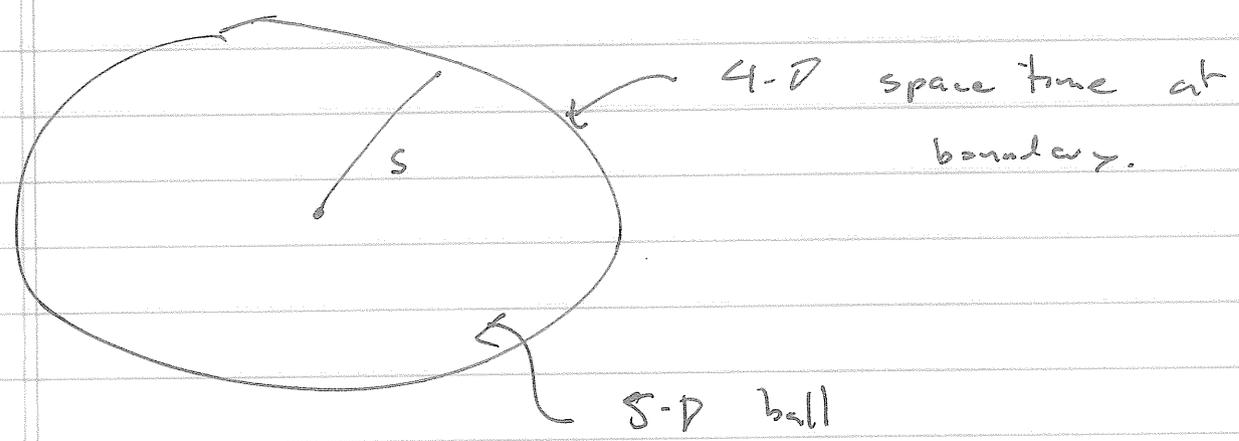
S_{WZ} (Wess-Zumino action)

$$= D \int_0^1 ds \Delta(s)$$

$$= i \int_0^1 ds \int d^4x \frac{E^{abcd}}{48\pi^2} \text{Tr} [\Sigma^a \partial_b \Sigma^c \partial_d \Sigma^e \partial_e \Sigma^f \partial_f \Sigma^g \partial_g \Sigma^h \partial_h \Sigma^i \partial_i \Sigma^j \partial_j \Sigma^k \partial_k \Sigma^l \partial_l \Sigma^m \partial_m \Sigma^n \partial_n \Sigma^o \partial_o \Sigma^p \partial_p \Sigma^q \partial_q \Sigma^r \partial_r \Sigma^s \partial_s \Sigma^t \partial_t \Sigma^u \partial_u \Sigma^v \partial_v \Sigma^w \partial_w \Sigma^x \partial_x \Sigma^y \partial_y \Sigma^z \partial_z \Sigma^4]$$

(multiply by N_c for N_c colours of quarks)

$$S_{WZ} = \frac{i N_c}{240\pi^2} \int d^5x \epsilon^{\mu\nu\rho\sigma\lambda} \text{Tr} [\Sigma^a \partial_\mu \Sigma^b \partial_\nu \Sigma^c \partial_\rho \Sigma^d \partial_\sigma \Sigma^e \partial_\lambda \Sigma^f \partial_f \Sigma^g \partial_g \Sigma^h \partial_h \Sigma^i \partial_i \Sigma^j \partial_j \Sigma^k \partial_k \Sigma^l \partial_l \Sigma^m \partial_m \Sigma^n \partial_n \Sigma^o \partial_o \Sigma^p \partial_p \Sigma^q \partial_q \Sigma^r \partial_r \Sigma^s \partial_s \Sigma^t \partial_t \Sigma^u \partial_u \Sigma^v \partial_v \Sigma^w \partial_w \Sigma^x \partial_x \Sigma^y \partial_y \Sigma^z \partial_z \Sigma^4]$$



(This interpretation due to Witten. Also explains the quantization of N_c .)