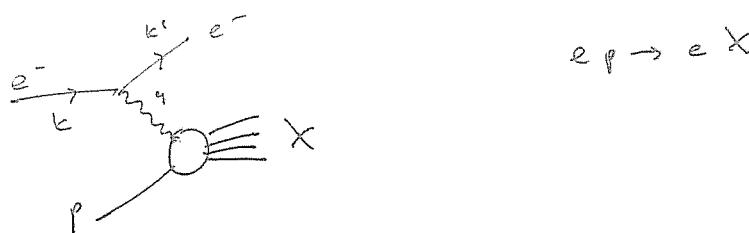


Operator Product Expansion & Deep Inelastic Scattering

DES

 $e p \rightarrow e X$

$$\cancel{q} = k - k' \quad q^2 = -2k \cdot k' = 2\gamma mc^2$$

$$= -2E E' (1 - \cos\theta)$$

$$= -Q^2$$

$Q^2 \gg \Lambda_{\text{QCD}}^2$ so QCD can be applied.

$$i\mathcal{M}(ep \rightarrow eX) = \bar{u}(k') (\not{v} \gamma^\mu) u(k) - \frac{i}{q^2} \int d^4x e^{-iq \cdot x} \langle X | J_\mu(x) | P \rangle$$

(Pestov has wrong sign in ~~phase factor~~ phase factor of Eq. 18.101)

$$\langle e^-(k') | \not{v} \gamma^\mu u(e(k)) \rangle \sim e^{-i(k-k') \cdot x} \bar{u}(k') \not{v}^\mu u(k)$$

$$\sim e^{-iq \cdot x} \bar{u}(k') \not{v}^\mu u(k)$$

see phase factor in definition of \not{v}, \not{u} (3.22, 3.100)

also. $\int d^4x e^{-iq \cdot x} \langle X | J_\mu(x) | P \rangle$

$$\int d^4x e^{-iq \cdot x} \langle X | e^{iP \cdot x} J_\mu(0) P^{-iP \cdot x} | P \rangle$$

$$\int d^4x e^{-i(q-p_x+p) \cdot x} \sim (2\pi)^4 \delta^4(q+p-p_x)$$

$i\mathcal{M}(ep \rightarrow eX) = i e^2 \bar{u}(k') \not{v}^\mu u(k) \frac{1}{q^2} \langle X | J_\mu(0) | P \rangle (2\pi)^4 \delta^4(q+p-p_x)$

(2)

χ_i for averaging over electron spin

$$d\sigma = \frac{1}{2} \frac{1}{2s} \sum \overline{m^2} \cdot (2a)^4 \delta^4(q+p-p_i) \frac{d^3 p_i}{(2a)^3 2p_i^0}$$

$$\begin{aligned} &= \frac{1}{2s} \frac{e^4}{(2a)^2} \cdot T \left[k \gamma^\mu k' \gamma^\nu \right] \frac{d^3 k'}{(2a)^3 2k'^0} \\ &\propto \sum_k (2a)^4 \delta^4(q+p-p_i) \langle p | J_\mu(k) | x \rangle \langle x | J_\nu(k) | p \rangle \end{aligned}$$

\sum_{all} - includes sum over all possible states

as well as $\frac{d^3 p_i}{(2a)^3 2p_i^0}$ for each particle in final stat

$$\frac{1}{2} T \left[k \gamma^\mu k' \gamma^\nu \right] = 2(k^\mu k'^\nu - g^{\mu\nu} k \cdot k' + k^\nu k'^\mu)$$

$$k' = k'^0 = |\vec{k}'|$$

$$\frac{d^3 k'}{(2a)^3 2k'^0} = \frac{2\pi \cdot dk' k'^2 dk' \cos\theta}{(2a)^3 \cdot 2k'}$$

Some variables

$$x = \frac{\omega^2}{2p \cdot q}$$

Bjorken x.

$$\underline{\omega^2 = xys}$$

$$y = \frac{2p \cdot q}{2p \cdot k}$$

In proton rest frame:

$$x = \frac{2k \cdot k' (1 - \cos\alpha)}{2m(k - k')}$$

$$y = \frac{2p \cdot q}{2p \cdot k} \sim \frac{k - k'}{k}$$

$$y = 1 - \frac{k'}{k}$$

$$\begin{aligned} S_2: (k \cdot p)^2 &= 2k \cdot p \sim p^2 \\ &= 2mk \quad (\epsilon \gg m) \end{aligned}$$

$$x = \frac{2k \cdot k' (1 - \cos\alpha)}{ys}$$

(2)

$$dx = \frac{2k|k|}{ys} dk \quad dy = \frac{dk}{k}$$

$$dk' k' dk' \cos = (k d\gamma) \cdot k' \frac{ys}{2k|k|} dk' \cos = \frac{dy dx ys}{2}$$

$$\frac{d^3 k}{(2\pi)^3 2k} = \frac{q\pi}{(2\pi)^3} dk' k' dk' \cos = \frac{q\pi}{(2\pi)^3 2} dy dx ys = \boxed{\frac{dy dx ys}{(2\pi)^2}}$$

$$d\sigma = \frac{\alpha^2 \gamma}{(2\pi)^2} \left(k_u k_v + k_v k_u - g_{uv} k \cdot k' \right) \\ \times \sum_x (2\pi)^4 \delta^{(4)}(q - p - p_x) \langle p | J^\mu(x) | x \rangle \langle x | J^\nu(x) | p \rangle$$

Define ~~Value~~ $\omega^{\mu\nu} = i \int d^4 x e^{i\gamma x} \langle p | J^\mu(x) J^\nu(x) | p \rangle$

By the optical theorem:

$$2 \text{Im } \omega^{\mu\nu} = \sum_x (2\pi)^4 \delta^{(4)}(q - p - p_x) \langle p | J^\mu(x) | x \rangle \langle x | J^\nu(x) | p \rangle$$

~~Also~~, $\sum_x \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right| = \sum_{\mu} \sum_{\nu} (ie)^2 \sum_x (2\pi)^4 \delta^{(4)}(q - p - p_x) \langle p | J^\mu(x) | x \rangle \langle x | J^\nu(x) | p \rangle$

$$= 2 \text{Im} \left(\text{---} \right) = 2 \text{Im} (ie)^2 \because \omega^{\mu\nu}(p, q)$$

(4)

can also see this by direct computation

$$\begin{aligned}
 W^{mn} &= i \int d^4x e^{i\vec{q} \cdot \vec{x}} \langle p | T J^m(x) J^n(0) | p \rangle \\
 &= \sum_x i \int d^4x e^{i\vec{q} \cdot \vec{x}} \langle p | J^m(x) | x \rangle \langle x | J^n(0) | p \rangle \theta(x^0) + \dots \\
 &= \sum_x i \int d^4x e^{i\vec{q} \cdot \vec{x}} e^{i\vec{p} \cdot \vec{x}} e^{-i\vec{p}_0 \cdot \vec{x}} \theta(x^0) \langle p | J^m(0) | x \rangle \langle x | J^n(0) | p \rangle + \dots
 \end{aligned}$$

$$i \int d^4x e^{i(\vec{q} \cdot \vec{x} + i\vec{p} \cdot \vec{x} - i\vec{p}_0 \cdot \vec{x})} \theta(x^0) = (2\pi)^3 \delta^3(\vec{q} + \vec{p} - \vec{p}_0) i \int_0^\infty dx_0 e^{i(\vec{q}^0 + \vec{p}^0 - \vec{p}_0^0) \cdot x_0}$$

$$\begin{aligned}
 i \int_0^\infty dx_0 e^{i\vec{q} \cdot \vec{x}_0} &\rightarrow i \int_0^\infty dx_0 e^{i(\alpha + i\beta)x_0} = \frac{i}{i(\alpha + i\beta)} e^{i(\alpha + i\beta)x_0} \Big|_0^\infty \\
 &= -\frac{1}{(\alpha + i\beta)}
 \end{aligned}$$

$$W^{mn} = \sum_x - \left(\frac{1}{\vec{q}_0 + \vec{p}_0 - \vec{p}_x^0 + i\beta} \right) (2\pi)^3 \delta^3(\vec{p} + \vec{q} - \vec{p}_x) \langle p | J^m(0) | x \rangle \langle x | J^n(0) | p \rangle$$

$$\frac{1}{x+i\beta} = P\left(\frac{1}{x}\right) - i\pi \delta(x)$$

$$\Rightarrow 2 \int_m W^{mn} = \sum_x (2\pi)^4 \delta^4(p + q - p_x) \langle p | J^m(0) | x \rangle \langle x | J^n(0) | p \rangle$$

+ ... terms? They give contribution $\propto \delta^4(q + p_x - p)$

$q^0 > 0 \quad p_x^0 > p^0 \quad (\text{at least one baryon in final state})$

So this term vanishes because it is not possible to satisfy

(5)

↓ lepton tensor
↓ hadron tensor

$$\frac{d\sigma}{dx dy} = \frac{q^2}{Q^2} \gamma \left(k_{\mu} k_{\nu}' + k_{\mu}' k_{\nu} - g_{\mu\nu} k \cdot k' \right) 2 \operatorname{Im} W^{\mu\nu}$$

To proceed further we must decompose $W^{\mu\nu}$

$\operatorname{g}_{\mu\nu} W^{\mu\nu} = 0$ because of current conservation.

$$W^{\mu\nu} = \left(-g^{\mu\nu} + q^{\mu} q^{\nu} \frac{1}{q^2} \right) W_1 + \left(p^{\mu} - q^{\mu} \frac{p \cdot q}{q^2} \right) \left(p^{\nu} - q^{\nu} \frac{p \cdot q}{q^2} \right) W_2$$

a function of p^{μ} , q^{μ} . Easily guessed, or just start w/
 $W^{\mu\nu}$. A sum of $B q^{\mu} q^{\nu} + C (p^{\mu} q^{\nu} + q^{\mu} p^{\nu}) + D p^{\mu} p^{\nu}$

$$\operatorname{g}_{\mu\nu} W^{\mu\nu} = 0 \Rightarrow \text{two eqns.}$$

Noting ~~glll pppk~~

$$q^{\mu} q^{\nu} \equiv (k_{\mu} k_{\nu}' + k_{\mu}' k_{\nu} - g_{\mu\nu} k \cdot k')$$

$$q^{\mu} \frac{1}{q} \operatorname{Tr} [k_{\mu} k_{\nu} k'_{\mu'} k'_{\nu'}]$$

$$= \frac{1}{q} \operatorname{Tr} [p_{\mu} p_{\nu} (k - k') k' k']$$

$$k' k = k^2 = 0$$

$$= \frac{1}{2} \operatorname{Tr} [(k^2 - k'^2) k' k] = 0.$$

$$-g^{\mu\nu} (k_{\mu} k_{\nu}' + k_{\mu}' k_{\nu} - g_{\mu\nu} k \cdot k') \equiv (4-2) k \cdot k'$$

$$= 2 k \cdot k'$$

$$= -q^2 = \omega^2 = 2 p \cdot q \propto$$

$$= \underline{XYS}$$

$$p^{\mu} p^{\nu} (k_{\mu} k_{\nu}' + k_{\mu}' k_{\nu} - g_{\mu\nu} k \cdot k') = 2 p \cdot k p \cdot k' = 2 p \cdot k p \cdot k - q^2 \approx \frac{(2 p \cdot k)^2}{2} \left(1 - \frac{2 p \cdot k}{2 p \cdot q} \right) = \underline{\frac{S^2(1-\gamma)}{2}}$$

(6)

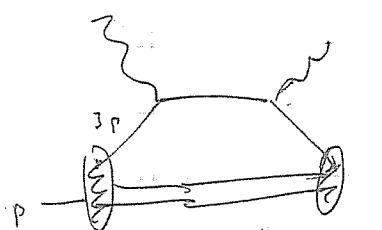
$$\frac{d\sigma}{dx dy} = \frac{\alpha^2 \gamma}{\alpha^4} \left[\frac{s^2(1-y)}{2} 2 \operatorname{Im} w_{12} + xy s 2 \operatorname{Im} w_1 \right]$$

so computing cross section boils down
computing structure function

$$2 \operatorname{Im} w_{12}$$

$2 \operatorname{Im} w_{12}$ function of P, q , scalars are $P^0, 2P \cdot q, \frac{q^2 = -Q^2}{x Q^2}$

Parton Model for F_1, F_2



parton distribution function

$$W^{uv} = i \int d^4x e^{iq \cdot x} \int_0^1 \frac{dz}{z} \sum_i f_{i/p}(z) \langle q(p) | T_r [J^u(x) J^v(z)] | q(z) \rangle$$

$$W^{uv} = i \int_0^1 \frac{dz}{z} \sum_i f_{i/p}(z) q_i^2 \bar{u}(p) \gamma^5 \frac{i(p+q)}{(p+q)^2 + m^2} \gamma^v u(p)$$

$$\text{Spin average} = \int_0^1 \frac{dz}{z} \sum_i f_{i/p}(z) q_i^2 \frac{i T_r [\delta^5(p+q) \gamma^v p]}{2p \cdot q + q^2 + m^2}$$

$$\text{To get } 2 \operatorname{Im} W^{uv} \text{ from } \frac{-1}{x + i\epsilon} = 2\pi \delta(x)$$

(7)

$$2 \operatorname{Im} \left(\frac{-1}{2p \cdot q - q^2 - i\epsilon} \right) = 2 \operatorname{Im} \frac{-1}{(2 \not{p} \not{q} - \not{q}^2 + i\epsilon)} \\ = 2\pi \delta(2 \not{p} \not{q} - \not{q}^2) \\ = 2\pi \delta \left(\frac{\not{q}^2}{x} \delta - \not{q}^2 \right) \\ = 2\pi \sum_{\alpha^2} \delta(\not{s} - \not{x}) - \frac{2\pi}{ys} \delta(\not{s} - \not{x})$$



does not contribute
in physical region

$$\frac{1}{2} T_a T_b g^{\mu\nu}(p_{\mu} q_{\nu}) \gamma^5 \gamma^5 = 2 \left[(p \cdot q)^{\mu} p^{\nu} + p^{\nu} (p \cdot q)^{\mu} - g^{\mu\nu} p \cdot (\overset{\circ}{p+q}) \right] \\ = 2(4 p^{\mu} p^{\nu} + 2(g^{\mu\nu} p^{\nu} + q^{\nu} p^{\mu}) - g^{\mu\nu} 2p \cdot q) \\ = 4 \not{s}^2 \not{p}^{\mu} \not{p}^{\nu} + 2 \not{s} (q^{\mu} \not{p}^{\nu} + q^{\nu} \not{p}^{\mu}) - g^{\mu\nu} 2 \not{s} \not{p} \cdot q \\ \rightarrow \underline{4 \not{x}^2 \not{p}^{\mu} \not{p}^{\nu} + 2 \times (p^{\mu} q^{\nu} + p^{\nu} q^{\mu}) - g^{\mu\nu} \times ys}$$

$$2 \operatorname{Im} W^{\mu\nu} = \sum_i q_i^2 f_{i/p}(x) \underset{xs}{=} \begin{cases} 4 \not{x}^2 \not{p}^{\mu} \not{p}^{\nu} + 2 \times (p^{\mu} q^{\nu} + p^{\nu} q^{\mu}) - g^{\mu\nu} \times ys \end{cases}$$

$$\left(p^{\mu} - q^{\mu} \frac{p \cdot q}{q^2} = p^{\mu} + \frac{q^{\mu}}{2x} \right) = \frac{1}{2x} \left[4 \not{x}^2 \left(p^{\mu} + \frac{q^{\mu}}{2x} \right) \left(p^{\nu} - \frac{q^{\nu}}{2x} \right) - \cancel{q^{\mu} q^{\nu}} - \cancel{g^{\mu\nu} \times ys} \right]$$

$$2 \operatorname{Im} W^{\mu\nu} = \sum_i q_i^2 f_{i/p}(x) \underset{xs}{=} \left[4 \not{x}^2 \left(p^{\mu} - q^{\mu} \frac{p \cdot q}{q^2} \right) \left(p^{\nu} - q^{\nu} \frac{p \cdot q}{q^2} \right) + \left(-g^{\mu\nu} + \frac{q^{\mu} q^{\nu}}{q^2} \right) \times ys \right]$$

$$2 \operatorname{Im} W_1 = 2\pi \sum_i q_i^2 f_{i/p}(x)$$

$$2 \operatorname{Im} W_2 = \frac{ys}{ys} \sum_i q_i^2 \times f_{i/p}(x)$$

(8)

Sometimes it is convenient to introduce $F_{1,2}$

$$2 \operatorname{Im} W_{\mu\nu} = \left(-g_{\mu\nu} + q_\mu q_\nu \frac{\alpha^2}{\pi^2} \right) F_1 + \frac{1}{p \cdot q} \left(P_\mu - q_\mu \frac{p \cdot q}{q^2} \right) \left(P_\nu - q_\nu \frac{p \cdot q}{q^2} \right) F_2$$

$$\begin{aligned} F_1 &= 2 \operatorname{Im} W \\ F_2 &= 2 \frac{\operatorname{Im} W_2}{p \cdot q} = \frac{\alpha^2}{2\pi} 2 \operatorname{Im} W_2 \\ &= \frac{\gamma_S}{2} 2 \operatorname{Im} W_2 \end{aligned}$$

$$\Rightarrow F_1(x, \alpha^2) = 2 \pi \sum_i q_i^2 f_{iN}(x)$$

$$\boxed{F_2(x, \alpha^2) = 2 \times F_1(x)}$$

Callan-Gross relation

Bjorken scaling: $F_{1,2}$ independent of α^2

→ indication of point-like substructure of proton

G-G - relation: partons - spin- $\frac{1}{2}$ particles

(4)

Applying OPE to

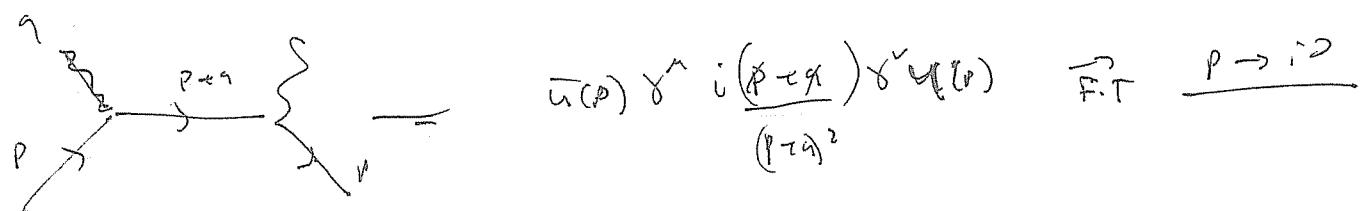
$$W^{uv} = \int d^4x e^{iq \cdot x} \langle [T | S^u(x) S^v(x)] | 0 \rangle$$

to lowest order in α_s

$$\bar{q} \gamma^{\mu} q(x) \bar{q} \gamma^{\nu} q(0)$$

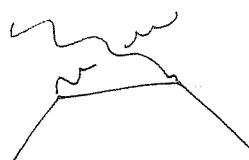
$$= \bar{q} \gamma^{\mu} q(x) \overbrace{\bar{q} \gamma^{\nu} q(0)} + \overbrace{\bar{q} \gamma^{\mu} q(x) \bar{q} \gamma^{\nu} q(0)}_{\textcircled{B}}$$

$$\int d^4x e^{iq \cdot x} \bar{q}(x) \overbrace{\gamma^{\mu} q(x) \bar{q}(0)} \gamma^{\nu} q(0) = \overbrace{\bar{q} \gamma^{\mu} i \frac{(i\cancel{p} - q)}{(i\cancel{p} + q)} q(0)}_{\textcircled{B}}$$



$$= \bar{q}(p) \gamma^{\mu} i \frac{(p - q)}{(p + q)^2} \gamma^{\nu} q(p) \quad \text{F.T} \quad \overbrace{p \rightarrow i\omega}$$

(4) corresponds to



$$\frac{1}{(i\cancel{p} + q)^2} = \frac{1}{q^2 + 2iq \cdot \omega - \omega^2} = \frac{-i}{\omega^2 + 2iq \cdot \omega + \omega^2} \approx \sum_n \frac{-1}{\omega^n} \left(\frac{2iq \cdot \omega + \omega^2}{\omega^2} \right)^n$$

We will give argument for dropping $i\omega$ below
essentially: $i\omega \sim 3p$ in perturbation model

$$\frac{2iq \cdot \omega}{\omega^2} \rightarrow \frac{2q \cdot 3p}{\omega^2} = \frac{3}{x} \sim O(1) \quad \text{where } \frac{\omega^2}{\omega^2} \sim \frac{3^2 p^2}{\omega^2} \ll 1$$

we always assume

$$\frac{q^2}{\omega} \ll 1 \quad (\text{as } \alpha \ll 1)$$

$$\frac{p}{\omega} \ll 1.$$

Dirac structure: $\gamma^m (\not{p} + \not{q}) \gamma^v$

we know that $w^{uv} = w^{vu}$ (this is only piece that contributes to $D_{\mu\nu} S$)

$$\gamma^m (\not{p} + \not{q}) \gamma^v \rightarrow \frac{1}{2} \left(\gamma^m (\not{p} + \not{q}) \gamma^v - \gamma^v (\not{p} + \not{q}) \gamma^m \right)$$

$$\gamma^m \gamma^2 \gamma^v = \underbrace{g^{\mu 2} \gamma^v - g^{\mu v} \gamma^2 + g^{\nu 2} \gamma^m}_{+ i \epsilon^{\mu \nu \nu \lambda} \gamma_5} + i \epsilon^{\mu \nu \nu \lambda} \gamma_5$$

$$\rightarrow (\not{p} + \not{q}) \gamma^v - g^{\mu v} (\not{p} + \not{q}) + (\not{p} + \not{q}) \gamma^m$$

recalling $w^{uv} = \left(-g^{uv} - \frac{g^{\mu u} g^{\nu v}}{g^{\mu \mu}} \right) w_1 + \left(p^m - g^m \frac{p \cdot q}{g^{\mu \mu}} \right) \left(\frac{p^v - q^v}{g^{\mu \mu}} \right) w_2$

$$\rightarrow -g^{\mu v} w_1 + p^\mu p^\nu w_2$$

because $(g^{\mu u} g^{\nu v}) L_{\mu\nu} = 0$ $L_{\mu\nu}$ = lepton tensor

we can drop γ^m, γ^v 's

Furthermore

$$\underline{i \not{p} q (0) = 0}$$

~~Dirac eq.~~ Dirac eq.

we obtain

$$\int d^4x e^{iq \cdot x} \bar{q}(x) \gamma^\mu q(x) \bar{q}(x) \gamma^\nu q(x)$$

$$= -i \bar{q} [\gamma^\mu (\not{D} + \gamma^\nu \not{D}^\mu - g^{\mu\nu} \not{g})] \sum_{\alpha^2} \left(\frac{2i q \cdot \alpha}{\alpha^2} \right)^n q$$

$$= -i \bar{q} [\gamma^\mu (\not{D} + \gamma^\nu \not{D}^\mu - g^{\mu\nu} \not{g})] \sum_{\alpha^2} \frac{(2q_m) \dots (2q_{mn})}{(\alpha^2)^n} \not{D}^\mu \dots \not{D}^m q$$

so general structure of local operator in QPE
is $\{ \text{sign} \rightarrow i \not{D}^n \text{ to make sense inv.} \}$

$$\bar{q} \gamma^{m_1} \not{D}^{m_2} \not{D}^{m_3} \dots \not{D}^{m_n} q$$

1) we can symmetrize in $m_1 \dots m_n$ (all are contracted into g_{mi})

2) first expansion consider

$$\langle p | \bar{q}_f \gamma^{m_1} \not{D}^{m_2} \dots \not{D}^{m_n} q_f | p \rangle$$

necessary two dimensions
for mass

$$= A_f p^{m_1} \dots p^{m_n} + A_f' g^{m_1 m_2} p^{m_3} \dots p^{m_n} (p^2) + \dots$$

True for dimensions

(f - labels fermion flavor.)

$$\frac{(2q_{m_1}) \dots (2q_{mn})}{(\alpha^2)^n} p^{m_1} \dots p^{m_n} = x^{-n} \sim O(1)$$

$$\frac{2q_{m_1} \dots 2q_{mn}}{(\alpha^2)^n} \frac{g^{m_1 m_2} p^{m_3} \dots p^{m_n} (p^2)}{(\alpha^2)^n} = x^{-n+2} \cdot \frac{m^2}{\alpha^2} \ll 1$$

ordinary OPE e.g. weak decay

$$\int dx e^{ix \cdot x} J_\mu(x) J_\nu(0) \sim \sum_a C_a(\omega^2) \partial_a$$

mass dimension = 2

\uparrow
 $\sim L$
 ω^{d-2}

mass dimension d

Monster in PDS importance of an operator controlled by twist not mass

$$\langle p | \int dx e^{ix \cdot x} j^m(x) j^\nu(0) | p' \rangle \sim \text{dimensionless}$$

$$\langle z | p' \rangle = 2 \pi \delta^3(p - p')$$

$$\sim \frac{2^m}{a} \dots \frac{2^m}{a} \frac{1}{\partial^{d-2}} \langle \partial^{m_1 \dots m_n} \rangle$$

$$\propto \frac{2^m \dots 2^m}{a^n} \frac{1}{\partial^{d-2}} \langle p^{m_1 \dots m_n} \rangle$$

$$\sim \left(\frac{1}{x}\right)^n \left(\frac{m_1}{a}\right)^{d-n-2}$$

$$\text{twist} = d - n \quad n - \text{spin}$$

$$J_\mu \dots J_\nu = \bar{q} [q, i \partial^m \dots i \partial^{n-1}] q$$

$$d = 3 + n - 1 = 2 + n$$

leading twist-2 operators $\left(\frac{m_p}{a}\right)^{n-1} \approx 0(1)$ higher twist $\sim \frac{m_p}{a}$ suppression

(14)

Some special operators:

$$n=1 \quad \langle p | \overline{q}_f \gamma^5 q_f | p \rangle = N_f 2 p^m$$

flavor current

Remember we are spin averages so

$$\langle p | \bar{q} \gamma^\mu q | p \rangle = \frac{1}{2} \sum_s \bar{u}_s(p) \gamma^\mu u_s(p) N_f$$

$$= \frac{1}{2} \bar{u}(p \gamma^\mu) N_f = N_f 2 p^m$$

$$\boxed{A_f^1 = N_f}$$

~~for~~
For
 $N_f = 2$ (up quarks)

$N_f = 1$ (down quarks)

$$n=2 \quad \langle p | \bar{q}_f \gamma^\mu i D^\nu \gamma^5 q_f | p \rangle$$

$$\langle p | T_f^{\mu\nu} | p \rangle$$

contribution to energy-momentum tensor from quark f

A_f^2 = momentum fraction carried
by quark flavor.

~~After~~ Unitary Residuality:

$$\sum_i A_i^2 \quad " \quad " \quad "$$

~ sum over all partons, including gluons

(13)

~~in terms of first 2 operators~~

we have evaluate $\bar{s} \gamma^{\mu} s(x) \bar{q} \gamma^{\nu} q(0)$

the other contraction is obtained by
 $x \leftrightarrow 0 \quad \Rightarrow \quad q \rightarrow -\bar{q}$ and $\mu \leftrightarrow \nu$.

this implies only even powers of q
appear in OPE.

$$\begin{aligned} & \int d^4x e^{iq \cdot x} j^{\mu}(x) j^{\nu}(0) \\ &= \sum_f Q_f^2 \left[4 \sum_{n=2}^{\infty} \frac{2q^{m_1} \dots 2q^{m_n}}{(e^2)^{n-1}} \partial_x^{m_1 m_2 \dots m_{n-2}} \right. \\ & \quad \left. - g^{\mu\nu} \sum_{n=0}^{\infty} \frac{2q^{m_1} \dots 2q^{m_n}}{(e^2)^n} \partial_x^{m_1 \dots m_n} \right] \end{aligned}$$

taking matrix elements $\langle p | \partial_x^{m_1 \dots m_n} | p \rangle$

$$= 2 A_f^{\mu\nu} p^{m_1} \dots p^{m_n}$$

we find

$$W^{\mu\nu} = \sum_f Q_f^2 \left[8 \sum_n p^{\mu} p^{\nu} \frac{(2q \cdot p)^{n-1}}{(e^2)^{n-2}} A_f^{nn} - 2 j^{\mu\nu} \sum_n \left(\frac{2q \cdot p}{e^2} \right)^n A_f^{nn} \right]$$

(15)

$$W_1 = \sum_f Q_f^2 \sum_n 2 \left(\frac{2q \cdot p}{\alpha^2} \right)^n A_f^n$$

$$W_2 = \sum_f Q_f^2 \sum_n \frac{g}{\alpha^2} \left(\frac{2p \cdot q}{\alpha^2} \right)^{n-2} A_f^{n-2}$$

$$\sum_n \frac{g}{\alpha^2} \left(\frac{\alpha^2}{2p \cdot q} \right)^2 \cdot \left(\frac{2q \cdot p}{\alpha^2} \right)^n A_f^n$$

$$F_1 = 2 \operatorname{Im} W_1 = \sum_f Q_f^2 \sum_n 2 \left(\frac{2q \cdot p}{\alpha^2} \right)^n A_f^n$$

$$F_2 = 2 \frac{\operatorname{Im} W_2}{p \cdot q} = \sum_f Q_f^2 \sum_n g \frac{\alpha^2}{2p \cdot q} \left(\frac{2q \cdot p}{\alpha^2} \right)^n A_f^n$$

$$\Rightarrow \underline{F_2(x, \omega^2) = 2 \times F_1(x, \omega^2)}^* \quad \text{Callan - Gross relation.}$$

Q. Relating OPE to DFS.

OPE assumption $\omega^2 \gg (\text{any other kinematic moment})$

reality $2p \cdot q \gg \omega^2$

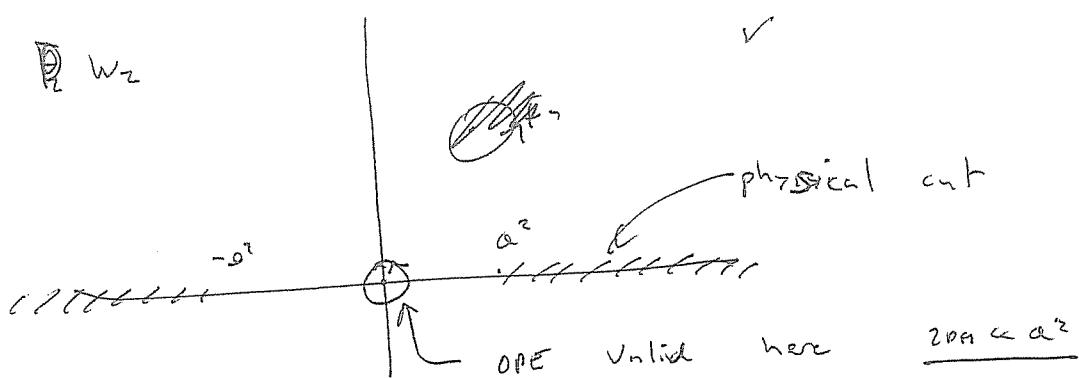
so what we have really shown is $\frac{2W_2}{p \cdot q} = 2 \times (2W_1)$ for

$2p \cdot q \ll \omega^2$, which is in the unphysical region. To relate the OPE to structure function, need dispersion relation to relate OPE to $\operatorname{Im} W_{1,2}$ for $2p \cdot q \geq \omega^2$ ($\omega \propto c$)

(16)

analytic structure of w_2

$$\text{let } v = 2\rho \cdot q = \frac{\omega^2}{\alpha^2} / x = \gamma s \\ = 2m_p E_\gamma \quad (\text{in proton rest frame})$$



Physical scattering $v > \alpha^2$ ($x \leq 1$)

Second cut is due to $\underline{w_2^{\mu\nu}(v, \alpha^2) = W(v, \alpha^2)}$
 $w_{1,2}(\bar{v}, \alpha^2) = w_2(-v, \alpha^2)$

This is easy to see if we note that
 ~~$w^{\mu\nu}$ is symmetric under $(\mu \leftrightarrow \nu, \mu \leftrightarrow v)$ see defn. p(3)~~

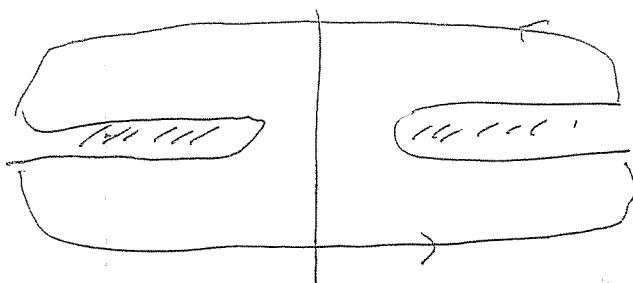
~~$w^{\mu\nu} = \delta^{\mu\nu} + \dots$~~

$$I_n = \int \frac{dv}{2\pi i} \frac{1}{v^{n+1}} w_2(v, \alpha^2)$$

$$= \int \frac{dv}{2\pi i} \frac{1}{v^{n+1}} \sum_j \frac{d^j}{\alpha^2} \left(\frac{v}{\alpha^2} \right)^{n-2} A_j^n =$$

$$= \sum_f \frac{A_f^n}{(\alpha^2)^{n+1}} A_f^n.$$

B Now let's run contours



B Contribution in this evaluation comes from discontinuities along branch cuts:

$$\Gamma_n = -2 \int_{\omega_1}^{\infty} \frac{1}{2\pi i} \frac{1}{x^{n-1}} 2i \operatorname{Im} w_2(v, \omega^2)$$

change variables

$$v = \frac{\omega^2}{x}$$

$$v = \frac{\omega^2}{x}$$

$$x = \frac{\omega^2}{\delta p \cdot s}$$

$$\text{B. } \frac{1}{(\omega^2)^{n-1}} \int_0^1 dx x^{n-2} \frac{v}{4\pi} \operatorname{Im} w_2$$

$$\boxed{f_f^+(x) = f_f^-(x) - f_f(x)}$$

from parton model

$$\rightarrow \boxed{x f_f^+(x, \omega^2) = \frac{xs}{4\pi} \operatorname{Im} w_2} = \frac{\omega^2}{4\pi} \operatorname{Im} w_2 = \frac{v}{4\pi} \operatorname{Im} w_2$$

$$\Gamma_n = \sum_f \frac{\omega_f^2}{(\omega^2)^{n-1}} A_f^n = \frac{1}{(\omega^2)^{n-1}} \int_0^1 dx x^{n-1} f(x)$$

$$\Rightarrow \boxed{A_f^n = \int_0^1 dx x^{n-1} f^+(x)} \leftarrow \begin{array}{l} \text{matrix elements of} \\ \text{local operators} \end{array}$$

= moments of p.d.f.s.

$$\text{B. } f^+(x, \omega^2) = f_+(x, \omega^2) - f_-(x, \omega^2)$$

(n even)

analysis of π pT's yields

$$A_f^n = \int_0^1 dx x^{n-1} f_f(x) \quad \underline{f_f = f_f(x) - \bar{f}_f(x)} \quad (18)$$

In parton model

$$\int_0^1 dx f(x) = N_f \quad \Leftrightarrow \quad A_f^1 = N_f$$

$$\int_0^1 dx x f(x) = \langle x \rangle_f \quad \rightarrow \quad A_f^2 \text{ is energy momentum carried by the parton } f.$$

$$\underbrace{\sum_f \langle x \rangle_f + \langle \bar{x} \rangle_f}_{\sum_f} = 1$$

These results are intuitive in ~~parton~~ parton model. consequence of

$$\langle p | \bar{q}_f q_f | p \rangle = 2 N_f p^m$$

$$A_f^1 = N_f$$
$$A_f^2 = 1$$

$$\langle p | T^{\mu\nu} | p \rangle = 2 p^\mu p^\nu$$

(19)

Violations of Scaling

we saw in parton model

$$F_{1/2}(x, \alpha^2) \Rightarrow F_{1/2}(x)$$

Our analysis of OPE for DSS shows that

$$\int_0^1 dx x^{n-1} f_f^\pm(x, \alpha^2) = A_f^n \quad \begin{array}{l} t \text{ is even} \\ \approx n \text{ odd} \end{array}$$

$$\text{where } \langle p | O_f^{(n)} \psi_1 \dots \psi_n | p \rangle = 2 A_f^n \rho^n \dots \rho^n - \text{traces}$$

Suppose (incorrectly) that $O_f^{(n)}$ obeyed homogeneous

$$\text{LRB: } \frac{d}{dn} O_f^{(n)} = -\gamma^{(n)} O_f^{(n)}$$

$$\gamma^{(n)} = \cancel{a_f} a_f^n \frac{g^2}{(4\pi)^2} + \dots$$

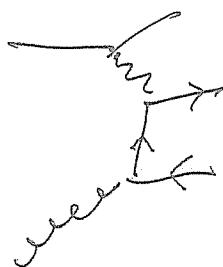
$$O_f^n(\alpha) = \left(\frac{\ln \alpha^2/\mu^2}{\ln (\epsilon^2/\mu^2)} \right)^{a_f^n/2\beta_0} O_f^{(n)}(\mu)$$

resumes logarithms of $\alpha_s(n) \ln(\frac{\alpha^2}{\mu^2})$,
gives leading corrections to Bjorken scaling

(20)

calculation of anomalous dimension is more complex, because of operator mixings

In LO gluons can't contribute to DSS but NLO they can:



Must also consider twist-2 gluon operators

$$O_g^{(n)} = -\frac{1}{2} F^{\mu_1 \nu} iD^{\mu_2} \dots iD^{\mu_n} F^{\nu \mu}$$

$$[O_g^{(n)}] = n+2 \quad (\text{n-1 derivatives and } \bar{\psi} \psi)$$

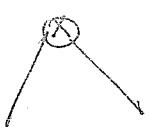
$$[O_g^{(n)}] = n+2 \quad \text{n-2 } \partial^\mu \text{'s} + 2 \bar{P}^\mu$$

these operators can mix.

Feynman rules for

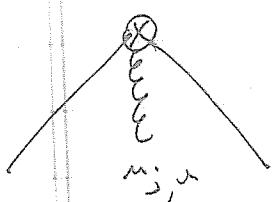
$$O_g^{(n)} = \bar{\psi} \gamma^{\mu_1} iD^{\mu_2} \dots iD^{\mu_n} \psi$$

$$= \delta^{\mu_1 \mu_2 \dots \mu_n} \text{ (Traces)}$$



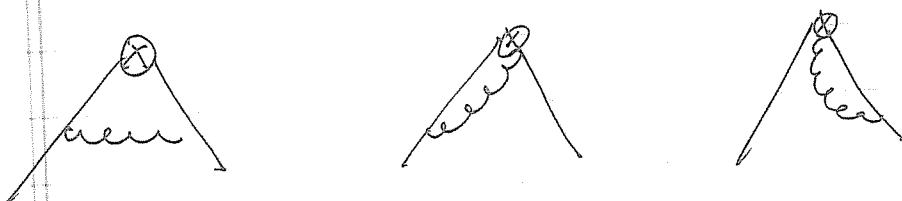
(21)

Because of covariant derivatives must run $g A^m$



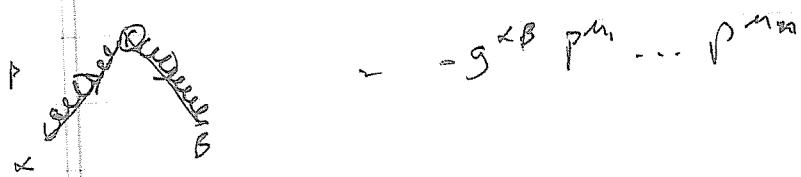
$$g T^a Y^M p^{M_1} \dots p^{M_n} \leftarrow \text{by omission.}$$

Diagram contributing to renormalization of $\mathcal{O}^{(n)}$



(For detailed evaluation see Peskin 18.5)

Box Gluon operator (n even)



$$- - g^{AB} p^{M_1} \dots p^{M_n}$$

Parallel

$$\frac{1}{2} F^{M_1} i D^{M_2} \dots i D^{M_{n-1}} P^{M_n}_{\nu} = \frac{1}{2} (\partial^M A^\nu - \partial^\nu A^M) i D^{M_2} \dots i D^{M_{n-1}} (\partial^M A_\nu - \partial_\nu A^M)$$

$$= \frac{1}{2} \partial^M A^\nu i D^{M_2} \dots i D^{M_{n-1}} \partial^M A_\nu$$

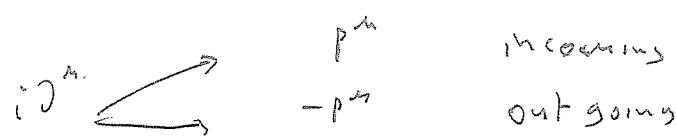
$$\sim \partial^M A^\nu \partial^M A_\nu \rightarrow \epsilon \cdot p = 0.$$

$$\partial^\nu A^M \partial_\nu A^M \rightarrow p^2 \epsilon^{M_1} \epsilon^{M_2} = 0$$

$$\rightarrow \frac{1}{2} \partial^M A^\nu i D^{M_2} \dots i D^{M_{n-1}} \partial^M A_\nu$$

$$= \frac{1}{2} i \partial^M A^\nu i \partial^{M_2} \dots i \partial^{M_{n-1}} i \partial^M A_\nu$$

(22)



$$\langle g_s(r) | -\frac{1}{2} i D^m A^r i D^m \dots i D^m A_r | \tilde{g}(r) \rangle$$

$$= \gamma^{ab} P^a \dots P^m$$

while opposite contraction differs by factor $(-1)^n$.

$$S_0 = \sum_{\text{odd } n} \gamma^{ab} P^a \dots P^m \quad (\text{even})$$

$$= \sum_{\text{odd } n} \quad \quad \quad (\text{odd})$$

more generally

$$F^{m_1 \nu} i D^{m_2} \dots i D^{m_{n-1}} F_{\nu}^{m_n}$$

(implied antisymmetrization/tracesless)

$$= i D^{m_1} (F^{m_2 \nu} \dots F_{\nu}^{m_n}) - i D^{m_2} F^{m_1 \nu} i D^{m_3} \dots i D^{m_{n-1}} F_{\nu}^{m_n}$$

repeating $n-2$ times

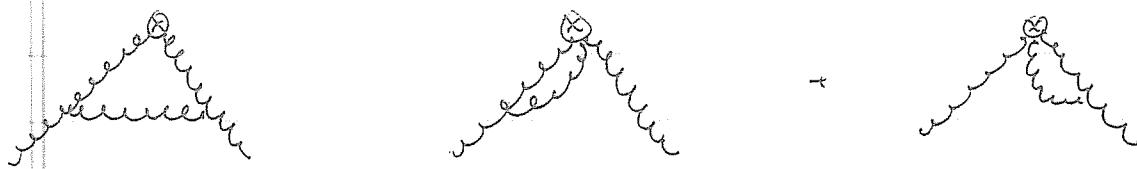
$$F^{m_1 \nu} i D^{m_2} \dots i D^{m_{n-1}} F_{\nu}^{m_n} = (-)^n i D^{m_1} \dots i D^{m_{n-1}} F^{m_1 \nu} F_{\nu}^{m_n} + \text{total derivatives}$$

$\partial_g^{(n)} = (-)^n \partial_g^{(n)}$ & total derivatives.

n odd: $2 \partial_g^{(n)} = \text{total derivatives}$

(24)

Finally must consider pure gluon graphs



Result for the anomalous dimension matrix

$$Y^n = \frac{-g^2}{(4\pi)^2} \begin{pmatrix} \alpha_{ff}^{nn} & \alpha_{fs}^{nn} \\ \alpha_{sf}^{nn} & \alpha_{ss}^{nn} \end{pmatrix}$$

$$\alpha_{ff}^{nn} = -\frac{8}{3} \left[1 + 4 \sum_{j=2}^n \frac{1}{j} - \frac{2}{n(n+1)} \right]$$

$$\alpha_{fs}^{nn} = 4 \frac{n^2 + n + 2}{n(n+1)(n+2)} n_f$$

$$\alpha_{sf}^{nn} = \frac{16}{3} \frac{n^2 + n + 2}{n(n^2-1)}$$

$$\alpha_{ss}^{nn} = -6 \left[\frac{1}{3} + \frac{2}{9} n_f + 4 \sum_{j=2}^n \frac{1}{j} - \frac{4}{n(n+1)} - \frac{41}{(n+1)(n+2)} \right]$$

$$\underline{n=2} \left[\frac{8}{3} \left(1 + 4 \sum_{j=2}^2 \frac{1}{j} - \frac{2}{2(3)} \right) \right] = \frac{8}{3} \left(1 + 2 - \frac{1}{3} \right) = \frac{8}{3} \cdot \frac{8}{3} = \frac{64}{9}$$

$$\frac{1}{3} + 4 \cdot \frac{2}{9} - \frac{4}{2 \cdot 3} - \frac{41}{2 \cdot 3} = \frac{1}{3} + 2 - 2 - \frac{41}{6} = 0$$

$n=2$

$$\begin{pmatrix} a_{ff}^2 & a_{fs}^2 \\ a_{sf}^2 & a_{ss}^2 \end{pmatrix} = \begin{pmatrix} -\frac{64}{9} & \frac{4}{3} u_f \\ \frac{64}{9} & -\frac{4}{3} u_s \end{pmatrix}$$

Note $(1 \ 1)$ is zero ^{left} eigenvector of this matrix. the linear combination

$$O_g^{(n) \mu\nu} + \sum_f O_f^{(n) \mu\nu} = -\frac{1}{2} F^{\mu\alpha} F^\nu_\alpha + \sum_f \bar{\psi}_f \gamma^\mu \gamma^\nu \psi_f$$

Note that.

$$\langle p | O_g^{(n) \mu\nu} | p \rangle = A_g^{(n)} 2 p^\mu p^\nu$$

$$\langle p | O_f^{(n) \mu\nu} | p \rangle = A_f^{(n)} 2 p^\mu p^\nu$$

$$\text{but } \langle p | T^{\mu\nu} | p \rangle = 2 p^\mu p^\nu \quad (\text{energy momentum})$$

$$\Rightarrow \underline{A_g^{(n)} + \sum_f A_f^{(n)} = 1}$$

in terms of p.d.f.

$$\int_0^1 dx x^{n-1} f_{g/p}(x) = A_g^{(n)}$$

$$\Rightarrow \int_0^1 dx \left(x f_{g/p}(x) + \sum_f x f_{f/p}(x) \right) = 1$$

expresses the idea $\sum \text{parton momentum} = \text{proton momentum}$.

our analysis shows that it is possible to calculate u (or even v) dependence of moments of p.d.f.

To ~~parallel~~)

These can be also written as an integro-differential equation ~~as~~ for the $f(x_i)$ themselves.

Let's write

$$\partial_t = \left(\partial_t - \frac{1}{n_f} \sum_f \partial_f \right) + \frac{1}{n_f} \sum_f \partial_f$$

flavor non-singlet

flavor flavor singlet

Let's do the same w/ the p.d.f.s

$$\left(f_{in}^{(x)} - \frac{1}{n_f} \sum_j f_{jn}^{(x)} \right) = \frac{1}{n_f} \sum_j f_{jn}^{(x)}$$

e.g. two flavors $f_{u/p} = \frac{1}{2} (f_{u/p} + f_{d/p}) = \frac{1}{2} (f_{u/p} - f_{d/p})$

clearly $\sum_j = 1$ so this linear combination can't mix w/ $f_{g/a}$ so flavor-non-singlet operators/structure function simply obey

$$\frac{d}{dx} \partial_n^{NS} = -\gamma_n \partial_n^{NS}$$

$$\boxed{\partial_n = -\frac{\gamma_n^2}{(4\pi)^2} \alpha_F^n}$$

(23)

$\langle \phi | \phi_m \rangle$

$$\begin{aligned}\langle p_1 | \phi(x) | p_2 \rangle &= \langle p_1 | e^{ip_1 \cdot x} \phi(0) e^{-ip_2 \cdot x} | p_2 \rangle \\ &= e^{i(p_1 - p_2) \cdot x} \langle p_1 | \phi(0) | p_2 \rangle\end{aligned}$$

$$\langle p_1 | \partial_m \phi(x) | p_2 \rangle = i(p_1 - p_2)_m e^{i(p_1 - p_2) \cdot x} \langle p_1 | \phi(0) | p_2 \rangle$$

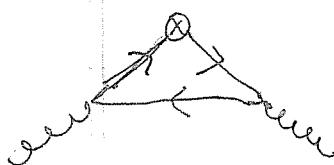
$$\Rightarrow \langle p_1 | \partial_m \phi(0) | p_2 \rangle = i(p_1 - p_2)_m \langle p_1 | \phi(0) | p_2 \rangle$$

$$\Rightarrow \underbrace{\langle p | \partial_m \phi(0) | p \rangle}_{} = 0$$

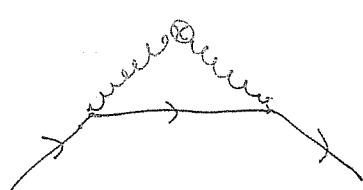
Since $\bar{F}^{m_1 n_1} iD^{m_2} \dots iD^{m_{n-1}} F^{m_n n}$ is a total derivative for n - odd it does not contribute to PDS.

~~Possible cases~~ the operator mixing n - even only.

Operator mixing comes from



$(\gamma_{fj}) \propto r_f$



(γ_{jj})

(27)

tRe

splitting - function

$$\boxed{n^2 \frac{d}{dx^n} f(x) = \frac{\alpha_s(n)}{2\pi} \int_x^1 dz \frac{f(z)}{z} P_{qg}(z) f^{ns}\left(\frac{x}{z}\right)} \quad (\text{AP})$$

Altarelli-Parisi Equation

$$P_{qg}(z) = \frac{4}{3} \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{z} \delta(1-z) \right]$$

tRe + distributions

$$\int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z) - f(0)}{1-z}$$

where $f(z)$ is assumed to be well-behaved as $z \rightarrow 1$.

We will see how to calculate P_{qg} directly later.
 Right now we just want to show AP equation is equivalent to our result for the moments.

$$\int_0^1 dx x^{n-1} \left\{ n^2 \frac{d}{dx^n} f^{ns}(x) = \frac{\alpha_s}{2\pi} \int_x^1 dz \frac{P_{qg}(z)}{z} f^{ns}\left(\frac{x}{z}\right) \right\}$$

~~$$\frac{d}{dx^n} f^{ns}(x) = \int_0^1 dz \frac{P_{qg}(z)}{z} f^{ns}\left(\frac{x}{z}\right) f^{ns}(z)$$~~

(27)

$$\int_0^1 dx \int_x^1 dz = \int_0^1 dz \int_0^z \theta(z-x) dx$$

$$= \int_0^1 dz \int_0^z dx$$

$$x = zu \quad \int_0^1 dz \int_0^z dx = \int_0^1 dz \int_0^1 du \cdot z$$

$$\text{so} \quad \int_0^1 dx \int_x^1 \frac{dz}{z} x^{n-1} P_{nq}(z) f^{ns}\left(\frac{x}{z}\right)$$

$$\int_0^1 du \int_0^1 dz (z \cdot u)^{n-1} P_{nq}(z) f^{ns}(u)$$

$$= \int_0^1 dz z^{n-1} P_{nq}(z) - \int_0^1 du u^{n-1} f^{ns}(u)$$

$$m^2 \frac{d}{du^2} \int_0^1 dx x^{n-1} f^{ns}(x) = \frac{\alpha_s}{2\pi} \left(\int_0^1 dz z^{n-1} P_{nq}(z) \right) \int_0^1 du u^{n-1} f^{ns}(u)$$

$$\frac{m^2 \lambda}{\lambda m^2} A_f^{(n)} = \frac{\alpha_s}{2\pi} \left(\int_0^1 dz z^{n-1} P_{nq}(z) \right) A_f^{(n)}$$

(2a)

$$\int_0^1 dz z^{n-1} \frac{4}{3} \left\{ \frac{1+z^2}{(1-z)_+} + \frac{z}{2} \delta(1-z) \right\}$$

~~$$\int_0^1 dz \frac{1}{(1-z)_+} = \int_0^1 dz \frac{4}{3} \left\{ \frac{z^{n-1}}{(1-z)_+} + \frac{z^{n-1}}{(1-z)_+} - \frac{z}{2} \delta(1-z) \right\}$$~~

$$\begin{aligned} \int_0^1 dz \frac{z^{n-1}}{(1-z)_+} &= \int_0^1 dz \frac{z^{n-1} - 1}{1-z} \\ &= \int_0^1 dz \frac{-(1-z)(1+z+z^2+\dots+z^{n-2})}{1-z} \\ &= - \int_0^1 dz (1+z+z^2+\dots+z^{n-2}) \\ &= - \sum_{j=1}^{n-1} \frac{1}{j} \end{aligned}$$

$$\int_0^1 z^j dz = \frac{1}{j+1}$$

$$\Rightarrow -\frac{4}{3} \left[\sum_{j=1}^{n-1} \frac{1}{j} + \sum_{j=1}^{n-1} \frac{1}{j} = \frac{3}{2} \right]$$

$$-\frac{1}{n} + \frac{1}{n+1} = \frac{-(n+1)+1}{n(n+1)} = -\frac{1}{n(n+1)}$$

~~$$\int_0^1 dz \frac{1}{(1-z)_+} = \sum_{j=1}^n \frac{1}{j}$$~~

$$= -\frac{4}{3} \left[\left(1 + \sum_{j=2}^n \frac{1}{j} - \frac{1}{n} \right) + \left(1 + \sum_{j=2}^n \frac{1}{j} + \frac{1}{n+1} \right) - \frac{3}{2} \right]$$

~~$$= -\frac{4}{3} \left[2 - \frac{3}{2} + 2 \sum_{j=2}^n \frac{1}{j} - \frac{1}{n(n+1)} \right]$$~~

$$\geq -\frac{2}{3} \left[1 + 4 \sum_{j=2}^n \frac{1}{j} - \frac{1}{n(n+1)} \right] \quad (n \geq 2)$$

$$\text{For } n=1 \quad \int_0^1 dz \frac{4}{3} \left[\frac{1+z^2}{(1-z)^2} + \frac{3}{2} \delta(1-z) \right]$$

$$\int_0^1 dz \frac{4}{3} \left(\frac{1+z^2 - z}{1-z} + \frac{3}{2} \delta(1-z) \right)$$

$$\left(\frac{z^2 - 1}{1-z} - (1-z) \right) = \frac{4}{3} \cdot \int_0^1 dz \left[- (1+z) + \frac{3}{2} \delta(1-z) \right]$$

$$= \frac{4}{3} \left\{ - \frac{3}{2} + \frac{3}{2} \right\} = 0.$$

$$\text{So } n^2 \frac{d}{dn^2} A_f^{NS} = \frac{ds}{2q} \left(-\frac{2}{3} \right) \left[1 + 4 \sum_{j=2}^n \frac{1}{j} - \frac{2}{n(n+1)} \right] A_f^{NS(n)}$$

$$= \frac{g^2}{(4\pi)^2} \cdot -\frac{4}{3} \left[1 + 4 \sum_{j=2}^n \frac{1}{j} - \frac{2}{n(n+1)} \right] A_f^{n, NS}$$

$$\text{and } \frac{d}{dn} A_f^{NS,n} = -\frac{g^2}{(4\pi)^2} \left(\frac{8}{3} \left[1 + 4 \sum_{j=2}^n \frac{1}{j} - \frac{2}{n(n+1)} \right] \right) A_f^{n, NS}$$

\uparrow
 $A_f^{n, NS}$

which is equivalent to our RG \bar{E} for $O_f^{NS,n}$

No anomalous dimension $n=1$ $\int_0^1 dx f_{in}(x) = \nu_E$