

## BRST invariance

Gauge Fixed YM action is:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2g} (\partial^\mu A_\mu^a)^2 + \partial^\mu \bar{c}^a D_\mu c^a + \bar{\psi} (i\bar{\gamma}^\mu \gamma_5) \psi$$

¶ There are several questions we would like to answer:

- renormalizability: before gauge fixing are actions was unique (P conserving  $\mathcal{L}$  consisting of dimension-4 operators that are gauge invariants). Now that we have introduced non gauge invariant terms, how do we know that new non-gauge invariant structures, e.g.  $(A_\mu^a A_\mu^a)^2$ , will be needed to renormalize the theory at higher orders in pert. theory.

- Unitary S-matrix  $S^\dagger S = 1$

We want  $\langle \alpha, \text{phys.} | B, \text{phys} \rangle = \sum_{\text{phys}} \langle \alpha, \text{phys} | S^\dagger | \gamma, \text{phys} \rangle \langle \gamma, \text{phys} | S | B, \text{phys} \rangle$

$|\alpha, \text{phys}\rangle$  denotes physical state e.g. transversely polarized boson, as opposed to longitudinally polarized boson, or ghost

How do we know that we can restrict sum on r.h.s. only to states that are physical.

- uniqueness: will  $S$  depend on  $\mathcal{G}[A_\mu^a]$ ?

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The answer to both these question can be addressed using BRST symmetry

let's introduce auxiliary field  $B^a$

$$S = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{g}{2} (B^a)^2 + B^a D^\mu A_\mu^a + D_a^c D_\mu c^a + \bar{\psi}(i\gamma^\mu - m)\psi$$

Since  $B^a$  is non-dynamical, can be eliminated using its equations of motion

$$B^a = -\frac{D^\mu A_\mu^a}{g}$$

and we recover the gauge-fixed action. Or we can think of this as a trivial gaussian path integration

$$\begin{aligned} & \int DB^a e^{i \int \frac{g}{2} B^a{}^2 + B^a D^\mu A_\mu^a} \\ &= \int DB^a e^{i \int \frac{g}{2} \left( B^a - \frac{D^\mu A_\mu^a}{g} \right)^2 - \frac{(D^\mu A_\mu^a)^2}{2g}} \\ &= \int DB^a e^{i \int \frac{g}{2} B^a{}^2 - i \int \frac{(D^\mu A_\mu^a)^2}{2g}} \\ &\quad \times e^{-i \int \frac{(D^\mu A_\mu^a)^2}{2g}} \end{aligned}$$

The BRST transformation is:

$$\delta A_\mu^a = \epsilon D_\mu c^a$$

$$\delta \psi = ig \epsilon c^a \gamma^a \psi$$

$$\delta c^a = -\frac{1}{2} g \epsilon f^{abc} c^b c^c$$

$$\delta \bar{c}^a = \epsilon B^a$$

$$\delta B^a = 0$$

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where  $\epsilon$  is a Grassmann variable

For a gauge transformation

$$\delta A_\mu = \frac{1}{g} D_\mu \alpha(x) \quad \delta \psi = i \alpha(x) t^a \psi$$

so far these fields BRST trans. is  
a gauge transformation  $\alpha(x) \rightarrow s \epsilon C^a(x)$

Note: product of two Grassmann's is a c-number  
invariance of  $-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi} (iD - m) \psi$  is automatic.

To see invariance of gauge fixed action under  
BRST, first note nilpotency of BRST, which  
means that two successive BRST transformations  
always gives zero. For example,

$$\begin{aligned} 1 \quad \delta_2 \delta_1 A_\mu^a &= \delta_2 (\epsilon_1 D_\mu c^a) \\ &= \delta_2 \epsilon_1 (D_\mu c^a + g f^{abc} A_\mu^b c^c) \\ &= \epsilon_1 \left\{ D_\mu \left( -\frac{g}{2} \epsilon_2 f^{abc} c^b c^c \right) + g f^{abc} \epsilon_2 (D_\mu c^b) c^c \right. \\ &\quad \left. + g f^{abc} A_\mu^b \left( -\frac{g}{2} \epsilon_2 f^{cde} c^d c^e \right) \right\} \end{aligned}$$

$$\begin{aligned} &\sim \epsilon_1 \epsilon_2 \left\{ -\frac{g}{2} f^{abc} D_\mu (c^b c^c) + g f^{abc} D_\mu c^b c^c + \right. \\ &\quad \left. + g^2 f^{abc} f^{bed} \sum_n A_\mu^b c^d c^e - g^2 f^{abc} f^{bed} A_\mu^b c^d c^e \right\} \end{aligned}$$

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$$0(5) \quad \text{term} < f^{abc} \left( -\frac{1}{2} 2_m(c^b c^c) + 2_m c^b c^c \right)$$

$$< = f^{abc} \left( -\frac{1}{2} 2_m c^b c^c - \frac{1}{2} c^b 2_m c^c + 2_m c^b c^c \right)$$

$$= f^{abc} \left( -\frac{1}{2} 2_m c^b c^c - \frac{1}{2} 2_m c^b c^c + 2_m c^b c^c \right) = \underline{\underline{0}}$$

$$0(5^2) < f^{asc} f^{bde} A_m^d c^e c^c - \frac{1}{2} f^{asc} f^{cde} A_m^b c^d c^e$$

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~~switch~~

$$= f^{adc} f^{bde} A_m^b c^e c^c \quad "$$

$$= f^{abd} f^{cbe} A_m^b c^e c^a - \frac{1}{2} f^{asc} f^{cde} A_m^b c^d c^e$$

$$= (f^{adc} f^{cbe} - \frac{1}{2} f^{abc} f^{cde}) A_m^b c^d c^e$$

$$= \frac{1}{2} (f^{adc} f^{cbe} - f^{ace} f^{cbe} - f^{abc} f^{cde}) A_m^b c^d c^e$$

$$= \frac{1}{2} (f^{adc} f^{cbe} + f^{ace} f^{cde} + f^{abc} f^{cde}) A_m^b c^d c^e$$

$\underline{0}$  by Jacobi identity.

$$\delta_2 \delta_1 c^a = \delta \epsilon_1 \cdot \left[ -\frac{1}{2} g f^{abc} c^b c^e \right] \quad \checkmark \quad \text{this sign b.c. } c' \epsilon_2 = - \epsilon_2 c'$$

$$= \epsilon_1 \epsilon_2 \left[ -\frac{g^2}{4} f^{abc} f^{bde} c^f c^e - \frac{g^2}{4} f^{abc} c^b f^{cde} c^d c^e \right]$$

$$= \epsilon_1 \epsilon_2 \frac{g^2}{4} (f^{abc} f^{bde} c^c c^d c^e - f^{abc} f^{cde} c^c c^d c^e)$$

$$= \epsilon_1 \epsilon_2 \frac{g^2}{2} f^{abc} f^{bde} c^c c^d c^e$$

using  $c^c c^d c^e = c^d c^e c^c = c^e c^c c^d$

$$= \epsilon_1 \epsilon_2 \frac{g^2}{2} (f^{abc} f^{bde} - f^{abd} f^{bec} - f^{abf} f^{bed}) c^c c^d c^e$$

= 0.

$$\delta_2 \delta_1 \gamma = \delta_2 (\epsilon_1 \cdot i_j c^a t^a \gamma)$$

$$= \epsilon_1 \epsilon_2 \left[ -\frac{i g^2}{2} f^{abc} c^b c^c t^a + g^2 c^a t^a c^b t^b \right] \gamma$$

$$c^a c^b t^a t^b = \frac{1}{2} c^a c^b [t^a, t^b] = \frac{i}{2} f^{abc} c^a c^b t^c$$

$$= \frac{i}{2} f^{abc} c^b c^c t^a$$

$$= \epsilon_1 \epsilon_2 \frac{g^2}{2} (-i f^{abc} + i f^{abc}) c^b c^c t^a \gamma$$

= 0

$$\delta_2 \delta_1 \bar{c}^a = \delta_2 \epsilon_1 B^a = \underline{0}$$

$$\delta_2 \delta_1 B^a = \underline{0}$$

Since BRST is a symmetry of the action  
 (we'll prove this below.) There must be  
 a conserved ~~charge~~<sup>charge</sup> which generates BRST.  
 call this operator  $Q$ .

$$\delta\phi = [\varepsilon a, \phi]$$

$$\text{Bosons: } \delta\phi = i[\varepsilon a, \phi] = \varepsilon [a, \phi]$$

$$\begin{aligned} \text{Fermions: } \delta\chi &= i[\varepsilon a, \chi] = \varepsilon a\chi - \chi a \\ &= \varepsilon \{a, \chi\} \end{aligned}$$

$$\delta\phi = i\varepsilon [a, \phi]_+ \quad [a, \phi]_\pm := \begin{cases} [a, \phi] & \text{for boson} \\ \{a, \phi\} & \text{for fermion} \end{cases}$$

$$S_2 S_1 \phi = \varepsilon_1 \varepsilon_2 i [Q [a, \phi]_\pm]_\mp$$

$$\begin{aligned} \text{boson: } \{Q, [a, \phi]\} &= \{Q, a\phi - \phi a\} \\ &= a(a\phi - \phi a) + (\phi a - a\phi)a \\ &= [a^2, \phi] = 0 \end{aligned}$$

likewise for a fermion

$$[a, \{a, \chi\}] = [a^2, \chi] = 0$$

$$[Q^2, \text{anything}] = 0 \Rightarrow \boxed{a^2 = 0}$$

↑ nilpotency

$$Q = Q^+$$

if we assign fields ghost number

$$B^i, A^\mu, \psi = 0 \quad c = -1 \quad \bar{c} = -1$$

then  $\alpha = -1$  which why  $\alpha^2 = \text{const.} (\neq 1)$   
not an option.

Nilpotent operators introduce an interesting  
grading on the Hilbert space:

not closed  $Q|\psi_1\rangle \neq 0$

closed  $Q|\psi_0\rangle = 0$  but  $|\psi_0\rangle \neq Q|\psi'\rangle$

exact (null)  $|\psi_2\rangle = \alpha|\psi_1\rangle \Rightarrow \alpha|\psi_2\rangle = \alpha^2|\psi_1\rangle = 0$ .

physical states  $\rightarrow |\psi_0\rangle$

$|\psi_2\rangle$  are called null states

since they have vanishing inner product.

$$\langle \psi_2' | \psi_2 \rangle = \langle \psi_2' | Q | \psi_1 \rangle = \langle \psi_1' | \alpha^2 | \psi_1 \rangle = 0.$$

$$\text{also } \langle \psi_1 | \psi_0 \rangle = \langle \psi_1 | \alpha(\psi_0) = 0$$

Finally there is an equivalence among physical  
states:

$$\underline{|\psi_0\rangle \sim |\psi_0\rangle + |\psi_2\rangle} \leftarrow \text{equivalent states}$$

$$\text{let } |\psi_0\rangle_a = |\psi_0\rangle + \alpha|\psi_1\rangle$$

$$\langle \psi_0' | \psi_0 \rangle_a = (\langle \psi_0' | + \langle \psi_1' | \alpha) (|\psi_0\rangle + \alpha|\psi_1\rangle)$$

$$= \langle \psi_0' | \psi_0 \rangle + \langle \psi_1' | \alpha | \psi_0 \rangle + \langle \psi_0' | \alpha | \psi_1 \rangle + \langle \psi_1' | \alpha^2 | \psi_1 \rangle$$

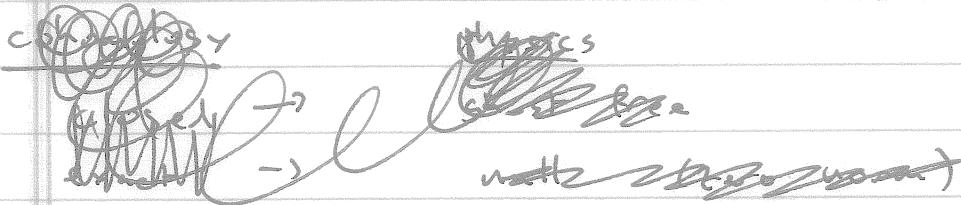
Hilbert space = cohomology of BRST operator

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we say Hilbert space is

closed ( $\alpha|u\rangle = 0$ )

exact ( $|u\rangle = \alpha|v\rangle$ )



How does this solve our unitarity problem?

$$\text{since } [\alpha, H] = 0 \quad [\alpha, S] = 0$$

i)  $\langle \text{ghost} | S | \text{phys} \rangle = 0$  (different)

$$\langle \text{ghost} | \alpha S | \text{phys} \rangle = \langle \text{ghost} | S \alpha | \text{phys} \rangle$$

" " "

$$g \langle \text{ghost} | S | \text{phys} \rangle = 0$$

if  $g \neq 0$  then  $\langle \text{ghost} | S | \text{phys} \rangle = 0$

ii)  $\langle \text{ghost} | (\alpha) S | \text{phys} \rangle = \langle \text{ghost} | \alpha S | \text{phys} \rangle$   
 $= \langle \text{ghost} | S \alpha | \text{phys} \rangle$   
 $= 0.$

$$\begin{aligned} \langle \langle, \text{phys} | \beta, \text{phys} \rangle &= \sum_n \langle \alpha, \text{phys} | S^\dagger | \cdot, n \rangle \langle n | S | \beta, \text{phys} \rangle \\ &= \sum_{\text{phys}} \langle \alpha, \text{phys} | S^\dagger | \beta, \text{phys} \rangle \langle \alpha, \text{phys} | S | \beta, \text{phys} \rangle \end{aligned}$$

because ~~with~~ null, ghost states don't contribute to the sum.

still have to show the gauge fixed action  
is invariant.

$$\text{let's introduce } \chi = \bar{c}^a \partial_m A^{am} + \frac{1}{2} \bar{c}^a B_a$$

$$\{\alpha, \chi\} = \{\alpha, \bar{c}^a\} \partial_m A^{am} - \bar{c}^a [\alpha, \partial_m A^{am}] +$$

$$+ \frac{1}{2} \{\alpha, \bar{c}^a\} B_a - \frac{1}{2} \bar{c}^a [\alpha, \overset{\circ}{B}_a]$$

$$= B^a \partial_m A^{am} - \bar{c}^a \partial_m D^m c^a + \frac{3}{2} B^a B_a$$

$$= B^a \partial_m A^{am} + \partial_m \bar{c}^a D^m c^a - \frac{3}{2} B^a B_a$$

which is nothing  $\mathcal{L}_{\text{GF}}$  expressed in terms  
of auxiliary field  $B_a$ . Evidently

$$\mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF}} = \mathcal{L}_{\text{YM}} + \{\alpha, \chi\}$$

so  $\alpha$  invariance of  $\mathcal{L}$  follows  
immediately from gauge invariance of  $\mathcal{L}_{\text{YM}}$   
and nilpotency of  $\alpha$  since

$$[\alpha, \{\alpha, \chi\}] = [\alpha^2, \chi] = 0$$

( $\chi$  is not a functional of the fields)  
 $A_m, \bar{c}, B^a \dots$

uniqueness imagine changing gauge  
fixing condition and interpret  
this as variation of functional  $\chi$

$$\chi \rightarrow \chi + \delta \chi$$

Then change in correlation function

$$\delta_x \langle 0 | O_1(x_1) O_2(x_2) \dots O_n(x_n) | 0 \rangle$$

$$= \langle 0 | \int d^4x \{ \partial_\mu \chi(x) \} \prod_i O_i(x_i) | 0 \rangle$$

$$= \langle 0 | \int d^4x \chi(x) \partial_\mu \prod_i O_i(x_i) | 0 \rangle$$

if  $O_i(x)$  are gauge invariant, ghost free  
operator

$$[Q, O_i(x_i)] = 0$$

$$\delta_x \langle \partial_\mu O_i(x_i) | 0 \rangle = 0 \quad \text{for gauge invariant observables.}$$

axial gauge:  $\bar{n}_\mu A^{\mu a} = 0 \quad 2 A^{\mu a} \rightarrow n_\mu A^{\mu a}$  in  $\chi$ .

$$\text{then } \underline{s \rightarrow 0} \quad \{ \partial_\mu \chi \} = B^\mu \bar{n}_\mu A^{\mu a} - \bar{c}^\mu \bar{n}_\mu \partial^\mu c^a$$

$B^\mu$  c.o.m set  $\bar{n}_\mu A^{\mu a} = 0$  + ghost

& ghost term is non interacting:  $-\bar{c}^\mu \bar{n}_\mu A^{\mu a} c^a$

we even use  $A_m^{\alpha}(x)$  for the operators in 56(A) provided the polarization vectors are transverse so the states are physical and  $\{a_i A_{\text{rhs}}^{\alpha i}\} = 0$ .

### Explicit example:

Explicit example QED in Lorentz gauge

YM

$$\delta A_m^\alpha = e \partial_m c^\alpha$$

$$\delta \psi = i y e c^\alpha t^\mu u$$

$$\delta c^\alpha = -\frac{1}{2} g f^{\alpha\beta\gamma} e f^{\gamma\delta\epsilon} c^\beta c^\epsilon$$

$$\delta \bar{c}^\alpha = e B^\alpha$$

$$\delta B^\alpha = 0$$

QED

$$\delta A_m^\alpha = e \partial_m c$$

$$\delta \psi = -i e e c^\alpha \psi$$

$$\delta c = 0$$

$$\delta \bar{c} = e B$$

$$\delta B = 0$$

L<sub>G.F.</sub><sup>QED</sup>

$$L_{\text{G.F.}}^{\text{QED}} = B \partial_m A^m + \frac{1}{2} B^2 + 2 \bar{c} \partial_m c$$

$$B = -\frac{e}{2} \partial_m A^m$$

let's ignore  $\psi$  (matter fields) & eliminate

$B$  using e.o.m. QED BRST transformations

are

$$\delta A_m^\alpha = e \partial_m c \quad \delta \bar{c} = -\frac{e}{2} \partial_m A^m \quad \delta c = 0$$

How does the operator  $a$  act on annihilation/creation operators in wave expansion of these fields?

$$\hat{A}_n(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2p^0}} \left( a_p^\mu e^{ip \cdot x} + a_p^{\mu*} e^{-ip \cdot x} \right)$$

Here  $a_p^\mu = \sum_{\lambda} a_{\lambda, p} \hat{\epsilon}_{\lambda}^\mu$  includes sum over phys. & unphysical polarization  $\lambda = 0, 1, 2, 3$

$$c(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2p^0}} \left( c_p e^{ip \cdot x} + c_p^* e^{-ip \cdot x} \right)$$

$$\bar{c}(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2p^0}} \left( b_p e^{ip \cdot x} + b_p^* e^{-ip \cdot x} \right)$$

Remember,  $c_p, b_p$  are distinct operators!

$$\delta A_n = i \epsilon [\alpha, A_n] \in \mathfrak{g}_{2n} c$$

$$\Rightarrow [\alpha, a_p^\mu] = p^\mu c_p \quad [\alpha, a_p^{\mu*}] = -p^\mu c_p^*$$

$$\delta \bar{c} = i \epsilon [\alpha, \bar{c}] = -\frac{\epsilon}{3} 2n A^\mu$$

$$\Rightarrow \{\alpha, b_p\} = -\frac{p_\mu a_p^\mu}{3} \quad \{\alpha, b_p^*\} = \frac{p_\mu a_p^{\mu*}}{3}$$

$$\delta c = i \epsilon [\alpha, c] = 0.$$

$$= \{Q, c\} = \{\alpha, c\} = 0$$

let  $|U\rangle$  be any physical state.

Now add a photon to this state:

$$|\mathcal{E}(p)U\rangle = \sum a_p^m |U\rangle$$

Under what conditions is this a physical state?

$$\begin{aligned} 1) \quad Q(\mathcal{E}(p), U) &= 0 \Rightarrow Q \sum a_p^{*m} |U\rangle \\ &= - [Q, \sum a_p^{*m}] |U\rangle \quad Q|U\rangle = 0 \text{ by assumption} \\ &= - \sum p^m c_p |U\rangle \\ &= 0. \end{aligned}$$

$$\text{So } Q(\mathcal{E}(p), U) = 0 \Rightarrow \boxed{\sum p^m = 0}$$

$$2) \text{ we also know } E^m \sim \varepsilon^m + \lambda p^m$$

$$\Rightarrow |\mathcal{E}(p)V\rangle = \sum a_p^{*m} |U\rangle \sim (\sum a_p^{*m} + \lambda \sum a_p^{*m}) |U\rangle$$

$$\sim |\mathcal{E}(p)U\rangle + \lambda \sum \{Q, b_p^*\} |U\rangle$$

$$\sim |\mathcal{E}(p)U\rangle + \lambda Q b_p^* |U\rangle$$

$$|\mathcal{E}(p)U\rangle \sim |\mathcal{E}(p)U\rangle + Q |U'\rangle \quad |U'\rangle \propto b_p^* |U\rangle$$

$$\boxed{\varepsilon^m \sim \varepsilon^m + \lambda p^m} \iff |\mathcal{E}(p)U\rangle \sim |\mathcal{E}(p)U\rangle + Q |U'\rangle$$

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Checking that there are no ghosts in phys Hilbert space

$$Q b^\dagger |v\rangle = \{a, b^\dagger\} |v\rangle = P_m a_p^m |v\rangle \neq 0$$

$$c^\dagger |v\rangle = - \frac{[a, \epsilon_m a_p^m]}{\epsilon_m \cdot p_m} = a - \frac{\epsilon_m a_p^m}{\epsilon \cdot p} |v\rangle = a |v'\rangle$$

$$\text{let } \epsilon^m(p) = (\epsilon_1^0, \epsilon_1^1, \epsilon_1^2, \epsilon_1^3) \quad p_m^m = (p, 0, 0, p)$$

$$= \epsilon_2(1021) +$$

$$\epsilon_1^m = \frac{(1001)}{\sqrt{2}} \quad \epsilon_2^m = (0, 1, 0, 0) \text{ or } (0, 0, 1, 0)$$

$$\epsilon_3^m = \frac{(100-1)}{\sqrt{2}} \quad \text{so } P_m \epsilon_1^m \neq 0 \quad \epsilon_3^m \propto p^m.$$

$$|v_0\rangle = \sum_{\lambda_m} \epsilon_{\lambda_m}^{a_p^m} |v\rangle \quad \text{transversely polarized photons}$$

$$|v_1\rangle = b^\dagger |v\rangle \quad \alpha |v_1\rangle \neq 0 \quad \text{antighost}$$

$$\sum_m a_p^m |v\rangle \quad \text{longitudinal boson}$$

$$|v_2\rangle \quad c^\dagger |v\rangle \quad v\rangle \sim a |v'\rangle \quad \text{ghost}$$

$$\sum_{\lambda_m} \epsilon_{\lambda_m}^{a_p^m} |v\rangle \quad \text{longitudinally polarized boson}$$

only physical, transversely polarized gauge bosons appear in  $|v_0\rangle$  which  
~~is~~ is the cohomology of the BRST operator  $a$ .