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PHY 342. Quantum Field Theory II.

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Textbook : Peskin and Schroeder, Intro. to QFT

Outline of the Course

Basically we are going to cover part III of P&S.

- Non Abelian gauge theory
- Quantization of NAGT
- QCD
- Operator product Expansion, Eff. Hamiltonians
- Anomalies
- Spontaneous Symmetry Breaking:

Goldstone's Theorem, chiral Perturbation Theory

Higgs Mechanism

- Quantization of ^{gauge theory} w/ SSB

These are basic ideas underlying
Standard Model of ~~Electrodynamics~~
Elementary Particle physics

Students are expected to know contents of

P&S Ch. 1-12: Free field theory, Perturbation theory,

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Non Abelian Gauge Theory - Motivation.

$$g_V = 1$$

$$g_A \approx 1.25$$

Weak Interactions

Fermi Theory of β -decay $n \rightarrow p e^- \bar{\nu}_e$

$$\mathcal{L}_{\text{Fermi}} = \frac{G_F}{\sqrt{2}} \bar{\psi}_p \Gamma_{up}^n \psi_n \bar{\psi}_e \Gamma_{un}^e \psi_e$$

$$G_F = 1.166 \cdot 10^{-5} \text{ GeV}^2$$

$$\delta(g_V = g_A \gamma_5)$$

seems

weird

$$P_m = \begin{cases} 1 & m(1-\gamma_5) \\ 0 & \text{otherwise} \end{cases}$$

$$[G_F] = \text{mass}^{-2} \quad - \text{non-renormalizable interaction}$$

consistent theory of weak interactions?

Current-current structure of $\mathcal{L}_{\text{Fermi}}$ suggest
following solution:

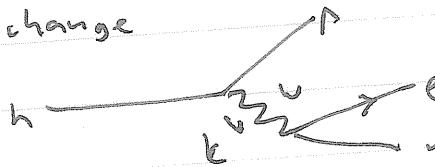
$$\mathcal{L}' = g W^m J_m^{up} + g W^m J_m^{er} \quad [g] = 0$$

hence gauge
theory

Where W^m is a massive vector boson

then the weak decay proceeds through MVB

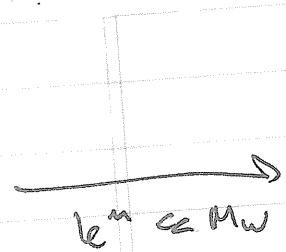
exchange



Feynman rules yield

$$\cdot (-ig)^2 \bar{u}(p) \Gamma_{up}^n u(n) \left[i(-g^{mn} + k^m k^n / M_W^2) \right] \bar{u}(p_e) \Gamma_{er}^e v(p_e)$$

$$k^2 - M_W^2 - i\epsilon$$



$$+ \frac{g^2}{M_W^2} \bar{u}(p) \Gamma_{up}^n u(n) \bar{u}(p_e) \Gamma_{er}^e v(p_e)$$

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identifying $\frac{g^2}{M_W^2} \sim \frac{e_F^2}{\Gamma^2}$ we see that this

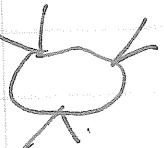
reproduces the Fermi theory at low energies ($k^4 \ll M_W$)

since $[W^\mu] = 1$ $[S^\mu] = 3$ the $[g] = 0$.

\mathcal{L}' is naively renormalizable. However, this is not quite correct as the arguments that led to this conclusion assume

$$\frac{e_F}{k} \sim \frac{1}{k^2} \quad \text{for large } k. \quad (\text{or } \frac{1}{k} \rightarrow \text{fermions})$$

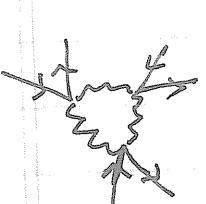
For example in ϕ^4 theory, no one-loop divergence in ϕ^6 amplitude



$$\sim \int d^4 e \frac{1}{(e^2)^3} \sim \text{UV finite.}$$

no ϕ^6 counterterm needed.

But W boson propagator $\rightarrow \frac{k^\mu k^\nu}{M^2} \frac{1}{k^2} \sim k^0$



$$\sim \int d^4 l \sim l^4.$$

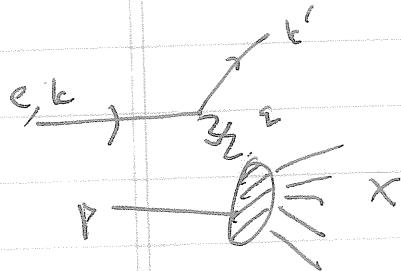
will need six-fermion operator.

Solution: if we give the vector bosons a mass through the Higgs mechanism we can obtain a gauge renormalizable gauge theory

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2nd motivation for NAGF

Deep Inelastic Scattering - SLAC ~ 1970



$$e p \rightarrow e' + X$$

$X = \text{anything}$

$$q^2 = -q^2_{\perp} = -2k \cdot k' = -2E E' (1 - \cos\theta) = -Q^2$$

More kinematic variables: P, q L.E. quantities

$$P^2 = 0 \quad \text{Req } q^2 = -Q^2 \quad 2P \cdot q = \frac{Q^2}{x_B} \quad \boxed{x_B = \frac{Q^2}{2P \cdot q}}$$

x - Bjorken variable (significance to appear below)

$$A \sim -ie \bar{u}(k') \gamma^\mu u(k) \underset{q^2}{\cancel{\langle X | J_{em}^\mu | P \rangle}}$$

$$\begin{aligned} & \text{do} \\ & \text{depends on } \sum_x S(P + q - p_x) \langle X | J_{em}^\mu | P \rangle (p_1 J_{em}^\nu / x) \\ & W^{\mu\nu}(P, q) - \text{hadronic tensor} \end{aligned}$$

$$W^{\mu\nu}(P, q) = W_1(x_B, Q^2) \left(-g^{\mu\nu} + \frac{q^{\mu\nu}}{Q^2} \right) + W_2(x_B, Q^2) \left(P^\mu - \frac{q^\mu P \cdot q}{Q^2} \right) \left(P^\nu - \frac{q^\nu P \cdot q}{Q^2} \right)$$

this follows from $q_\mu W^{\mu\nu}(P, q) = q_\nu W^{\mu\nu}(P, q) = 0$.

(and Ward identity for matrix element of J_{em}^μ)

(in) $W_1(x_B, Q^2), W_2(x_B, Q^2)$ - structure functions

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For $q^2 > \text{a few GeV}$

$$W_{1,2}(x, \alpha^2) \rightarrow W_i(x)$$

become independent of α^2 .
Bjorken Scaling

Parton model pretend that proton is collection

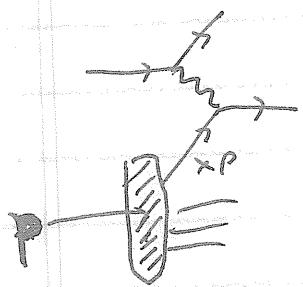
of non-interacting partons

Each parton carries fraction x

of proton of parton momentum
w/ prob. $f_i(x)$. (un known)

One can show ~~Bjorken~~ $x = x_B$.

$$+ \text{Im } W_{1,2}(\alpha^2) \propto \sum_i f_i(x)$$



Sum over i is sum over distinct parton

Species.

\Rightarrow partons are non-interacting in collision w/ large

momentum transfers ($\alpha^2 \gg \text{GeV}$) but clearly
strongly interacting at low momentum transfer

(confinement). This is property of asymptotic
freedom which is unique to non-abelian gauge
the-

Quantum Chromodynamics. non-abelian gauge theory
of quarks & gluons

predict non-trivial small ($\ln \alpha^2$) deviations

from Bjorken scaling in DIS

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Non-Abelian gauge theory.

$$L_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\cancel{D} - m)\psi \quad \cancel{D} = \partial + i\vec{e} A$$

is invariant under a local symmetry transformation

$$1) \psi \rightarrow e^{i\phi(x)} \psi \quad 2) A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \theta$$

clearly $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant

To see invariance of $\bar{\psi} i\cancel{D} \psi$

$$\begin{aligned} D_\mu \psi &= \partial_\mu \psi + i e A_\mu \psi \rightarrow \partial_\mu \left(e^{i\phi(x)} \psi \right) + i e \left(A_\mu - \frac{1}{e} \partial_\mu \theta \right) \psi \\ &= e^{i\phi(x)} \left[\partial_\mu \psi + i \partial_\mu \theta \psi + i e A_\mu \psi - i \partial_\mu \theta \psi \right] \\ &= e^{i\phi(x)} \left(\partial_\mu + i e A_\mu \right) \psi \\ &= e^{i\phi(x)} D_\mu \psi \end{aligned}$$

D_μ - covariant derivative $D_\mu \psi$ does not transform in a simple way under gauge transformation. We add a gauge field, A_μ , to the ordinary derivative to obtain a derivative operator which acts on ~~as well~~ ~~as follows that~~ has the property:

$$(\psi, D_\mu \psi) \rightarrow e^{i\phi(x)} (\psi, D_\mu \psi)$$

Local gauge symmetry.

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D_μ is a building block of gauge theory

$$[D_\mu, D_\nu] \psi \rightarrow e^{i\phi(r)} [D_\mu, D_\nu] \psi$$

$$[D_\mu, D_\nu] = [2_\mu + ieA_\mu, 2_\nu + ieA_\nu] = ie [2_\mu A_\nu - 2_\nu A_\mu] \\ = ie F_{\mu\nu}$$

so field strength is commutator of covariant derivatives.

To obtain Yang-Mills theory we generalize
to multicomponent fields

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}$$

Demand local invariance under

$$\psi_i \rightarrow U_{ij} \psi_j$$

or suppressing indices $\psi \rightarrow u\psi$, where it is understood that u is a matrix and ψ a vector

~~field is a complex valued field, it is~~
natural to demand ~~U~~ be unitary:

~~electromagnetism~~

let ψ be complex valued field, then

$$\psi^+ \rightarrow \psi^+ u^+$$

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clearly we want U to be unitary, then

$$L = \bar{U} (i\partial - m) U \rightarrow \bar{U} U^* (i\partial - m) U U^* \\ = \bar{U} (i\partial - m) U$$

* we will also demand
 $\text{Det } U = 1$,
 $U \in SU(n)$

provided we can construct covariant derivative D_m .

$$\partial_m \Psi \rightarrow \partial_m (U \Psi) = U \partial_m \Psi + \partial_m U \Psi \\ = U (\partial_m \Psi + U^* \partial_m U \Psi)$$

$$-ig A_m \Psi \rightarrow -ig A'_m U \Psi = U (-ig U^* A_m^* U) \Psi$$

$$\text{we want } D_m \Psi = (\partial_m - ig A_m) \Psi \rightarrow$$

$$\rightarrow U D_m \Psi = U (\partial_m - ig A_m) \Psi \\ =$$

$$\Rightarrow -ig \cancel{U^* A_m^* U} = -ig A_m + \cancel{U^* \partial_m U} \quad -ig U^* A'_m U + U^* \partial_m U = -ig A_m \\ \cancel{U^* A_m^* U} = A_m - \frac{i}{g} \cancel{U^* \partial_m U} \quad U^* A'_m U = A_m - \frac{i}{g} U^* \partial_m U$$

$$\cancel{A'_m} = U A_m U^* + \frac{i}{g} (\partial_m U) U^*$$

$$A'_m = U A_m U^* - \frac{i}{g} \partial_m U U^*$$

$$\text{gives } D_m = \partial_m - ig A_m \quad A_m \rightarrow A'_m = U A_m U^* - \frac{i}{g} \partial_m U U^*$$

note that since $U U^* = 1$ $\partial_m (U U^*) = (\partial_m U) U^* + U \partial_m U^* = 0$

$$\text{so } \partial_m U U^* = -U \partial_m U^*$$

$$A_m \rightarrow U A_m U^* + \frac{i}{g} U \partial_m U^*$$

(This is how Penrose writes it.)

(a)

Now that we have D_μ we can find $F_{\mu\nu}$

$$F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu]$$

$$= \frac{i}{g} [2_\mu - i g A_\mu, 2_\nu - i g A_\nu]$$

$$F_{\mu\nu} = 2_\mu A_\nu - 2_\nu A_\mu - i g [A_\mu, A_\nu]$$

- last term is non-vanishing because
 A_μ , ϵ and $F_{\mu\nu}$ are $n \times n$ matrices.

Note: $[D_\mu, D_\nu] \not\propto u [D_\mu, D_\nu] \propto$
 $u [D_\mu, D_\nu] u^+ u \propto$

or $[D_\mu, D_\nu] \rightarrow u [D_\mu, D_\nu] u^+$ i.e. $F_{\mu\nu} \rightarrow u F_{\mu\nu} u^+$

so $F_{\mu\nu}$ is not gauge invariant. We make
 gauge invariant by taking traces. e.g. $T_\mu [F_{\mu\nu} F^{\mu\nu}]$

To get Yang-Mills Lagrangian we write down
 most general gauge invariant Lagrangian
 w/ dimension 4 operator

$$\mathcal{L} = -\frac{1}{2} T_\mu [F_{\mu\nu} F^{\mu\nu}] - \kappa \epsilon^{\mu\nu\rho\sigma} T_\mu [F_{\mu\nu} F_{\rho\sigma}]$$

$$+ \bar{q} (\not{D} - m) q$$

mass dimensions

$$[F_{\mu\nu}] = 2 \quad [A_\mu] = 1 \quad [q] = \frac{3}{2}$$

Comments

- Also can consider complex scalar fields, just make derivatives covariant

$$\mathcal{L} = \bar{\mathbf{D}}^a \phi^* D_a \phi - m^2 \phi^* \phi$$

$$\mathbf{g} = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$$

- no mass for vector bosons $-\frac{1}{2}m^2 \text{Tr}[(A_\mu A^\mu)]$ is not gauge invariant

$$- L_{\text{gauge}} = -\frac{1}{2} \text{Tr}[F_{\mu\nu} F^{\mu\nu}]$$

kinetic term

$$= -\frac{1}{2} \text{Tr}[(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)]$$

$$- \frac{1}{2} \text{Tr}[(\partial_\mu A_\nu - \partial_\nu A_\mu)[A^\mu, A^\nu] + \frac{1}{4} \text{Tr}[[A^\mu, A^\nu][A_\mu, A_\nu]]]$$

cubic, and quartic terms gauge boson self-interactions



not present in QED
(at tree level).

(1)

Review of Ward Identity (qed) (response to student question)

consider ^{on-shell} amplitude which involves an external photon w/ momentum k^μ

$$\epsilon^\mu(k) \text{ diagram} = \sum_i(k) m_i(k)$$

where $m_i(k)$ represents all other factors from the Diagram. QED ward identity states

$$[c^\mu m_i(k)] = 0$$

(on-shell S-matrix elements)

Let's show this for Compton scattering:

$$\begin{array}{l} k^\mu \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} p^\mu \quad \begin{array}{l} k'^\mu \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} p'^\mu = \bar{u}(p') (-ie\gamma^\mu) \cdot \frac{i}{p+k-m} (-ie\gamma^\nu) u(p) \sum_i(k) \epsilon'_i(k)$$

$$= -ie^2 \bar{u}(p') \not{\epsilon}' \frac{1}{p+k-m} \not{\gamma}^\nu u(p)$$

$$\begin{array}{l} k^\mu \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} p^\mu \quad \begin{array}{l} k'^\mu \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} p'^\mu = \bar{u}(p') (-ie\gamma^\mu) \frac{i}{p'-k-m} (-ie\gamma^\nu) u(p) \sum_i(k) \epsilon'_i(k)$$

$$= -ie^2 \bar{u}(p') \not{\epsilon} \frac{1}{p'-k-m} \not{\gamma}^\nu u(p)$$

$$\text{sum} = -ie^2 \bar{u}(p') \left[\not{\epsilon}' \frac{1}{p+k-m} \not{\epsilon} + \not{\epsilon} \frac{1}{p'-k-m} \not{\epsilon}' \right] u(p)$$

(12)

now make replacement $\not{e} \rightarrow k$

$$\text{Diagram with } k^m \text{ and } \not{k} = -ie^2 \bar{u}(p') \left[\not{e}' \frac{1}{\not{p} + \not{k} - m} \not{k} + \not{k} \frac{1}{\not{p} - \not{k} - m} \not{e}' \right] u(p)$$

$$\text{but } \bar{u}(p') \not{k} \frac{1}{\not{p} - \not{k} - m} = \bar{u}(p') [\not{k} - \not{e}' + m] \frac{1}{\not{p} - \not{k} - m} (\bar{u}(p')(\not{p}' - m) = 0) \\ = -\bar{u}(p')$$

$$+ \frac{1}{\not{p} + \not{k} - m} \not{k} u(p) = \frac{1}{\not{p} + \not{k} - m} (\not{p} + \not{k} - m) u(p) = u(p)$$

$$\text{so } = -ie^2 \bar{u}(p') \left[\not{e}' - \not{e}' \{u(p)\} \right] = 0 \quad \checkmark$$

The Ward identity can actually be proven diagrammatically by considering fermion lines (or closed fermion loops) w/ arbitrary # of photon attachments. (7.4 Peskin & Schroeder)

W.I. is useful for:

- checking calculations of QED amplitudes
- simplifying polarization sums

$$\text{let } d\sigma \propto \sum_{\lambda} \hat{\epsilon}(\lambda) \hat{\epsilon}^{*\nu}(\lambda) M_\mu(\lambda) M_\nu^{*(\lambda)}$$

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$$(E, p_1, p_2, p_3)$$

$$\text{let } k^a = k(1, 0, 0, 1)$$

$$\text{so } k^a m_a(k) = 0 \Rightarrow k(m_0(k) + m_3(k)) = 0$$

$$\epsilon^a(\pm) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$$

$$\Rightarrow \sum_{\pm} \epsilon^{(\pm)} \epsilon^{\nu}(\pm) m_a(k) m_a^*(k) = |m_1(k)|^2 + |m_2(k)|^2$$

$$\text{But } -g^{\mu\nu} m_a(k) m_a^*(k) = -|m_0|^2 + |m_1|^2 + |m_2|^2 + |m_3|^2 \\ = |m_1|^2 + |m_2|^2$$

$$= \sum_{\pm} \epsilon^{(\pm)} \epsilon^{\nu}(\pm) m_a(k) m_a^*(k)$$

$$\Rightarrow \boxed{\sum_{\pm} \epsilon^a(x) \epsilon^{\nu}(\pm) \rightarrow -g^{\mu\nu}}$$

greatly simplifies
QED calculations

Physics behind W.I.

let us consider a Lorentz boost that leaves k_a

$$\text{invariant } \Lambda^a_b k_b = k_a$$

(This is called "little group" of k_a).

$$A^a_v = S_v + \omega^a_v \quad w^a_v = -\omega^a_v$$

$$w^a_v = \frac{1}{c} n \cdot \partial^a \chi / (\gamma c^2 \partial^0 \chi) + \omega^a_v + w$$

for any 4 vector $v^\mu v_\mu = \eta^{\mu\nu} v_\mu v_\nu = \text{const}$

$$v_\mu \rightarrow \Lambda_m^\alpha v_\alpha \Rightarrow \underline{\eta^{\mu\nu} \Lambda_m^\alpha \Lambda_\nu^\beta = \eta^{\alpha\beta}}$$

for infinitesimal boost $\Lambda_m^\alpha = \delta_m^\alpha + \omega_m^\alpha$

$$\eta^{\mu\nu} \Lambda_m^\alpha \Lambda_\nu^\beta = \eta^{\alpha\beta} + \omega^{\alpha\beta} + \omega^{\beta\alpha} = O(\omega^2)$$

$$\Rightarrow \underline{\omega^{\alpha\beta} = -\omega^{\beta\alpha}}$$

$$\underline{\omega_m^\alpha = -\omega_\alpha^\alpha}$$

$$\omega^{\alpha\beta} = \eta^{\alpha\beta} \omega_\alpha^\beta$$

Now consider $\Lambda_m^\alpha k_\alpha = k_m \Rightarrow \omega_m^\alpha k_\alpha = 0$

$$\text{for } k_m = k \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad k^\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \epsilon_m^\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$w_{m\alpha} = \begin{pmatrix} 0 & a & b & 0 \\ -a & 0 & 0 & -a \\ -b & 0 & 0 & -b \\ 0 & a & b & 0 \end{pmatrix} \quad w_m^\alpha k_\alpha = w_{m\alpha} k^\alpha = 0$$

$$w_{m\alpha} \epsilon_m^\alpha = w_m^\alpha \epsilon_m^\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \pm i \\ -i \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \pm \frac{ib}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ b \end{pmatrix}$$

$$\boxed{\Lambda_m^\alpha \epsilon_m^\alpha = (1 \mp i\theta) \epsilon_m^\alpha + \frac{a \pm ib}{\sqrt{2}} k_m^\alpha} \quad \hat{k}_m = \frac{k_m}{|k|}$$

Evidently: ϵ_m^α are not true four-vectors
 $(v_\mu \rightarrow \Lambda_m^\alpha v_\alpha)$

but transform inhomogeneously w/ inhomogeneous term $\propto k^\alpha$

(15)

To get a Lorentz invariant S-matrix must not only require invariance under $\underline{\epsilon} \rightarrow \lambda \nu \underline{\epsilon}_v$, must also require invariance under $\underline{\epsilon} \rightarrow \underline{\epsilon} + \lambda \underline{k}_n$ arbitrary λ .

The photon field operator

$$A_n(x) = \int \frac{d^3 k}{2\omega_k} \sum \left(\epsilon_n(z, k) a_{z,k} e^{ik \cdot x} + \epsilon_n^*(z, k) a_{z,k}^\dagger e^{-ik \cdot x} \right)$$

is not a four-vector but transforms as

$$A_n \rightarrow \lambda_\mu^\nu A_\nu + \partial_\mu \lambda$$

So we see invariance under $A_n \rightarrow A_n + \partial_\mu \lambda$

is required if one is to obtain L.I.

S-matrix for massless spin-one particles.

(Weinberg)

Since in QED $L_{\text{int}} = \int d^4 x j^\mu A_\mu$

invariance under $A_n \rightarrow A_n + \partial_\mu \lambda$

~~requires~~ requires A_n couple to

conserved current

$$\boxed{\partial_\mu j^\mu = 0}$$

so W.I. is really consequence of current conservation

For example, the Compton amplitude can be written as

$$A \propto \langle e^- | T \left[d^u x A_\mu j^\mu(x) \right] \left[M_y A_\nu j^\nu(y) \right] | e^- \rangle$$

$$= e^- \left(d^u x d^v y e^{-ik \cdot x} e^{ib \cdot y} \epsilon^u(k) \epsilon^v(k) \right) \langle e^- | T j_\mu(x) j_\nu(y) | e^- \rangle$$

replacing ~~$\epsilon^u(k)$~~ $\epsilon^u(k) \rightarrow k_\mu$ and using

$$\int d^u x k_\mu e^{-ik \cdot x} \langle \dots j^\mu(x) \dots \rangle$$

$$= i \int d^u x \partial_\mu e^{-ik \cdot x} \langle \dots j^\mu(x) \dots \rangle$$

$$= -i \int d^u y e^{-ib \cdot y} \langle \dots \partial_\mu j^\mu(x) \dots \rangle + \dots$$

The ... represents terms which do not contribute to the S-matrix, so if $\partial_\mu j^\mu = 0$ the the on-shell amplitude vanishes if $\epsilon_\mu(k) \rightarrow k_\mu$

Current conservation also gives interesting constraints on off shell ~~correlation~~ quantities

For this we need:

$$\frac{\partial}{\partial x^\mu} T[A(x) B(y)] = \sum_{x^\mu} \Theta(x^\mu - y^\mu) A(x) B(y) + \Theta(y^\mu - x^\mu) B(y) A(x) + T\left[\frac{\partial A(x)}{\partial x^\mu}, B(y)\right]$$

$$- T\left[\partial A(x), B(y)\right] + \delta(x^\mu - y^\mu) T[A(x), B(y)]$$

Applying this to $\langle 0 | T \hat{j}^n(x) \psi(y) \bar{\psi}(z) | 0 \rangle$

$$\stackrel{?}{=} \langle 0 | T \hat{j}_n(x) \psi(y) \bar{\psi}(z) | 0 \rangle = \langle 0 | T [2_n \hat{j}^n(x) \psi(y) \bar{\psi}(z)] | 0 \rangle$$

$$+ \delta(x^0 - y^0) \langle 0 | T [\hat{j}^0(x), \psi(y)] \bar{\psi}(z) | 0 \rangle$$

$$+ \delta(x^0 - z^0) \langle 0 | T [\psi(z) [\hat{j}^0(x), \bar{\psi}(z)]] | 0 \rangle$$

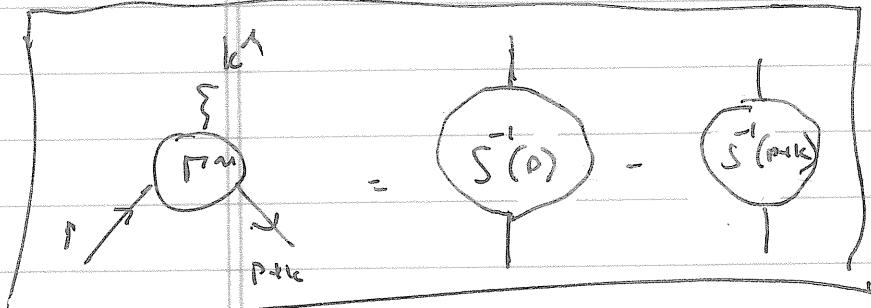
$$[\hat{j}^0(x), \psi(y)] = -e \gamma^0 \delta^3(x-y)$$

$$[\hat{j}^0(x), \bar{\psi}(z)] = e \bar{\psi}(z) \delta^3(x-z)$$

$$\stackrel{?}{=} \langle 0 | T \hat{j}_n(x) \psi(y) \bar{\psi}(z) | 0 \rangle = -q \delta^n(x-z) \langle 0 | T \psi(x) \bar{\psi}(z) | 0 \rangle + q \delta^n(x-z) \langle 0 | T \psi(y) \bar{\psi}(z) | 0 \rangle$$

after Fourier transform $\int_{-\infty}^{\infty} S(p) \hat{F}^n(p, p+k) S(p+k) = -S(p+k) + S(p)$

$$\boxed{\int_{-\infty}^{\infty} \hat{F}^n(p, p+k) = S^{-1}(p) - S^{-1}(p+k)}$$



This equation is useful for establishing relation

between electron's momentum & charge normalization

Lie Algebras

Earlier we have seen the following definition for N&G

$$D_\mu = \partial_\mu - i g A_\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g [A_\mu, A_\nu]$$

$$\mathcal{L}_M = -\frac{1}{2} \text{Tr} [F_{\mu\nu} F^{\mu\nu}] + \bar{\psi} (i \not{D} - m) \psi + \epsilon^{\mu\nu\rho\sigma} \text{Tr} [F_{\mu\nu} F^{\rho\sigma}]$$

Aside for now $\not{D} \rightarrow 0$. this term is a total derivative and violate P+T invariance. No The transformation properties of $F_{\mu\nu}$ under P,T are analogous to that of E&M

$$\begin{aligned} P: E_i &\rightarrow -E_i & \Rightarrow & F_{0i} \rightarrow F^{0i} \\ B_i &\rightarrow B_i & F_{ij} &\rightarrow F^{ij} \end{aligned} \quad \left. \begin{array}{l} F_{\mu\nu} \rightarrow F^{\mu\nu} \end{array} \right\}$$

$$\begin{aligned} T: E_i &\rightarrow E_i & \Rightarrow & F_{0i} \rightarrow -F^{0i} \\ B_i &\rightarrow -B_i & F_{ij} &\rightarrow -F^{ij} \end{aligned} \quad \left. \begin{array}{l} F_{\mu\nu} \rightarrow -F^{\mu\nu} \end{array} \right\}$$

$$\begin{aligned} F_\mu F^{\mu\nu} &\xrightarrow[P,T]{} F^{\mu\nu} F_{\mu\nu} & \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} &\xrightarrow[PT]{} \epsilon^{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \\ &&&= - \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta} \end{aligned}$$

Also a total derivative, in E&M

$$\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = 4 \epsilon^{\mu\nu\alpha\beta} \partial_\mu A_\nu \partial_\alpha A_\beta = 4 \partial_\mu (\epsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta)$$

(also a total derivative in \mathcal{L}_M , but can have terms)

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Transformations

$$A_\mu \rightarrow U A_\mu U^\dagger + i \frac{g}{\bar{g}} d_\mu U^\dagger$$

$$F_{\mu\nu} \rightarrow U F_{\mu\nu} U^\dagger \quad \mathcal{H} \rightarrow U \mathcal{H}$$

In our example $U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ $u_i, A_\mu, F_{\mu\nu}$ are $SU(n)$ matrices

 $SU(n)$:

$$U = e^{i\theta_a T^a}$$

$$\text{Det } U = 1 \Rightarrow \text{Tr } U^\dagger U = 0$$

($\ln \text{Det } M = \text{Tr } \ln M$ for any M)

$$U^\dagger U = 1 \Rightarrow e^{-i\theta_a T^a} e^{i\theta_a T^a} = 1 - i(T^a + T^a) \theta^a = \dots$$

$$T^a \theta^a = \sum T^a \theta^a$$

$$\Rightarrow T^a = T^{a\dagger} \quad T^a: \text{Traceless Hermitian generators of } SU(n)$$

(1)

$$[T^a, T^b] = i f^{abc} T^c$$

- structure constants

$$\text{real } [T^a, T^b]^\dagger = -[T^a, T^b] = -i f^{abc} T^c$$

This follows from CBH $U_1 U_2 = U_3$ (product of two group elements is also in the group - closure)

$$e^{i\theta_a T^a} e^{i\theta_b T^b} = e^{i(\theta_a T^a + \theta_b T^b + \underbrace{\theta_a \theta_b [T^a, T^b]}_{\text{must be linear combination of } T^c \text{'s}})/2 + \dots}$$

$$e^A e^B = e^{A+B+[A,B]/2 + \dots}$$

(2) Jacobi Identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0$$

writing this out, e.g.

$$[T^a, [T^b, T^c]] = T^a T^b T^c - T^b T^c T^a - T^a T^c T^b + T^c T^b T^a$$

There are 12 terms, every possible ordering of {a,b,c} appears twice w/ opposite sign.

representation of Lie Algebra -

set of matrices that realizes (1) + (2)

It is customary to choose a basis for these matrices so that

$$\text{Tr}[T_r^a T_r^b] = c_r \delta^{ab} \quad (3)$$

subscript r denotes representation, c_r can be different for different representations.

$$(3) \Rightarrow \underbrace{\text{Tr}[[T^a, T^b] T^c]}_{i c_r} = f^{abc}$$

f^{abc} are completely antisymmetric since

$$\begin{aligned} \text{Tr}[(T^a, T^b) T^c] &= \text{Tr}(T^a T^b T^c - T^b T^a T^c) \\ &= \text{Tr}[T^b T^c T^a - T^c T^b T^a] \\ &\geq \text{Tr}[[T^b, T^c] T^a] \\ &= \text{Tr}[(T^c, T^a) T^b] \end{aligned}$$

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Another important constraint on f^{abc} comes from (2)

$$[T^a [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0$$

$$if^{bcd} [T^a, T^d] + (a \leftrightarrow b \leftrightarrow c) = 0$$

$$- f^{bcd} f^{ade} f^{eef} + \dots = 0$$

$$\boxed{f^{ade} f^{bad} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0}$$

This allows us to find the adjoint representation

$$(T_g^a)_{bc} = -i f^{abc}$$

$$- (T_g^a)_{de} (T_g^b)_{cd} + (T_g^b)_{de} (T_g^a)_{cd} - i f^{abcd} (T_g^d)_{ce} = 0$$

$$\Rightarrow \boxed{([T_g^a, T_g^b])_{ce} = i f^{abd} (T_g^d)_{ce}} \quad \checkmark$$

Given a representation, t_r^a , $t_r^a = -t_r^{a*} = -t_r^{a\top}$

$$(\text{adjoint}) \text{ exists } ([t_r^a, t_r^b])^* = (i f^{abc} t_r^c)^*$$

$$[-t_r^{a*}, -t_r^{b*}] = i f^{abc} (-t_r^{c*})$$

t_r^a is conjugate representation

(22)

A representation may be distinct from its conjugate, or the conjugate maybe equivalent up similarity transformation

$$t_{\bar{r}}^a = S t_r^a S^{-1} \quad (1)$$

If so representation is real (or pseudoreal)

- $(T_g^a)_{bc}^* = -i \delta_{abc} = (T_g^a)_{bc}$ adjoint is real

for real representations it is possible to choose T_g^a to be pure imaginary (as in adjoint rep.) so $e^{i\alpha T_g^a}$ give group elements are real.

For pseudoreal reps. it is not possible to arrange this, but t_r and $t_{\bar{r}}$ are still equivalent. Example $\frac{\sigma^i}{2}$ generate $SU(2)$

$$\left[\frac{\sigma_i^k}{2}, \frac{\sigma_j^l}{2} \right] = i \epsilon_{ijk} \frac{\sigma^l}{2}$$

$$\sigma^i = -\sigma^2 \sigma^{i*} \sigma^2$$

so this is pseudo-real
(no basis in which all Pauli matrices are real.)

real S in (1) above is symmetric

pseudoreal S in (1) " " antisymmetric

(see Georgi, 259-262)

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adjoint of $su(2)$

$$(t^i)_{jkl} = -i \epsilon_{ijk}$$

$$J_1 = \begin{pmatrix} 0 & & \\ & 0 & i \\ & -i & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ -i & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} -i & & \\ i & 0 & \\ & & 0 \end{pmatrix}$$

$$\hat{x} \leftrightarrow \begin{pmatrix} 1 & \\ 0 & 0 \end{pmatrix} \quad \hat{y} \leftrightarrow \begin{pmatrix} 0 & \\ 0 & 0 \end{pmatrix} \quad \hat{z} \leftrightarrow \begin{pmatrix} 0 & \\ 0 & 1 \end{pmatrix}$$

J_1 generates rotation about \hat{x} -axis, etc.

This rep. is clearly real.

$d(g)$ -dim. of adjoint rep.

= # of Hermitian generators of $su(n)$

$$= N^2 - 1$$

Casimir Op.

$$T^\alpha T^\alpha$$

$$\begin{aligned} [T^\alpha T^\beta, T^\gamma] &= [T^\alpha, T^\beta] T^\gamma + T^\alpha [T^\beta, T^\gamma] \\ &= i f^{\alpha\beta\gamma} (T^\gamma T^\alpha + T^\alpha T^\gamma) = 0 \end{aligned}$$

$T^\alpha T^\alpha \propto \mathbb{1}$ for any rep.

$$\boxed{t_\alpha^\alpha t_\beta^\alpha = C_2(\gamma) \mathbb{1}} \quad \xrightarrow{\hspace{1cm}} \text{Casimir for irrep}$$

adjoint:

$$(t^c_{ab})_{ad} (t^c_{ab})_{db} = C_2(\gamma) \delta_{ab}$$

$$- f^{cad} f^{cda} = C_2(\gamma) \delta_{ab}$$

Casimir
for adjoint

$$\boxed{f^{acd} f^{bcd} = C_2(\gamma) \delta^{ab}}$$

$$\text{for } \text{SU}(2) \quad t^a = \frac{\sigma^a}{2} \quad \text{Tr}[t^a t^b] = \frac{1}{2} \delta^{ab}$$

Weyl-Mann matrices

$$\text{SU}(3) = \frac{1}{2} \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \frac{1}{2} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$x^{1,2,3} \qquad \lambda^4 \qquad \lambda^5 \qquad \lambda^6 \qquad \lambda^7$$

$$\lambda^8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\text{Since for any } \text{SU}(n) \quad \frac{1}{2} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$$

convention

$$\boxed{\text{Tr}[t_r^a t_b^b] = \frac{1}{2} \delta^{ab}}$$

$$\boxed{C_2(n) = \frac{1}{2} \quad \text{fundamental}}$$

$$\text{ex)} \quad t_r^a t_r^a = C_2(n) \mathbb{I}$$

$$\left(\text{Tr}[t_r^a t_p^b] \right)_r = (n) \delta^{ab} \quad \delta^{ab}$$

$$\text{Tr}[t_r^a t_r^a] = C_2(r) d(r)$$

$$\Rightarrow \boxed{C_2(r) d(r) = (r) d(r)}$$

Summary

$$d(r) = \text{dim. of irrep.}$$

$$\lambda(h) = \dots \text{ "adjoint" } = \# \text{ of generators}$$

$$N^2 - 1$$

$$C_2(r) = \text{quadratic Casimir of irrep.}$$

$$C(\mu) = \text{normalization of generators}$$

fundamental

$$C_2(n) d(n) = (n) d(n)$$

$$\Rightarrow C_2(n) N = \frac{1}{2} (N^2 - 1)$$

$$C_2(n) = \frac{N^2 - 1}{2n}$$

$$t^a t^a = \frac{N^2 - 1}{2n} \mathbb{I}$$

adjoint

$$C_2(s) = C(r) = N$$

(See next pg.) fundamental = adjoint $\otimes^N \text{SU}(N)$ (or see Peskin's argument.)

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useful identity for fundamental rep.

$$M_{ij} = 2 t_{ij}^a T_a [t^a M] + \frac{\delta_{ij}}{N} T_a [M]$$

(from completeness for Hermitian matrix, M)

$$M_{jk} \delta_{ik} \delta_{jr} = 2 t_{ij}^a t_{rk}^a M_{jk} + \frac{\delta_{jk}}{N} \delta_{ik} \delta_{jr} M_{jk}$$

$$\Rightarrow \delta_{ik} \delta_{jr} - \frac{1}{N} \delta_{ijkj} = 2 t_{ij}^a t_{rk}^a$$

$$\Rightarrow \boxed{t_{ij}^a t_{rk}^a = \frac{1}{2} \left(\delta_{ik} \delta_{jr} - \frac{1}{N} \delta_{ijkj} \right)}$$

$$\text{(*) } (t^a t^a)_{ij} = \frac{N^2 - 1}{2N} \delta_{ij}$$

$$T_a [t^a t^a] = \frac{N^2 - 1}{2} \quad \text{from which we infer } \text{tr} [t^a t^b] = \frac{1}{2} g^{ab}$$

Calculating $f^{abc} f^{abd} = C(G) \delta^{ad}$
 $\Rightarrow f^{abc} f^{abc} = C(G)(N^2 - 1)$

$$\text{Tr}[[T^a, T^b][T^c, T^d]] = i f^{abc} f^{bad} \text{Tr}[T^c T^d] \quad T^b - \text{fundamental}$$

$$= \frac{f^{abc} f^{abc}}{2}$$

$$\begin{aligned} \text{Tr}[[T^a, T^b][T^c, T^d]] &= 2 \text{Tr}[T^a T^b T^c T^d] - 2 \text{Tr}[T^a T^b T^d T^c] \\ &= 2 \text{Tr}[T^a T^c T^b T^d] - 2 \text{Tr}[T^a T^b T^c T^d] \end{aligned}$$

$$\text{Tr}[T^a T^c T^b T^d] = \left(\frac{N^2 - 1}{2N}\right)^2 \text{Tr}[1] = \frac{(N^2 - 1)^2}{4N}$$

$$\begin{aligned} \text{Tr}[T^a T^b T^c T^d] &= T_{ij}^a T_{jk}^b T_{ki}^c T_{ei}^d \\ &= T_{jk}^b T_{ki}^d \frac{1}{2} \left(S_{ie}^j S_{jk}^i - S_{ij}^k S_{ke}^i \right) \\ &= -\frac{1}{2N} \text{Tr}[T^b T^d] = -\frac{1}{2N} \frac{N^2 - 1}{2N} \cdot N \\ &= -\frac{N^2 - 1}{4N} \end{aligned}$$

$$\Rightarrow \text{Tr}[[T^a, T^b][T^b, T^a]] = 2 \frac{(N^2 - 1)^2 + N^2 - 1}{4N} = 2 \frac{N^2 - 1}{4N} \left(N^2 - (-1) \right) = \frac{N^2 - 1}{2} N$$

$$\Rightarrow f^{abc} f^{abc} = N(N^2 - 1)$$

or $\boxed{C(G) = N}$

$f^{abc} f^{abd} = N \delta^{cd}$
$f^{abc} f^{abc} = N(N^2 - 1)$

transformations, covariant derivatives for fundamental
vs. adjoint representations

Fundamental $\psi \rightarrow U\psi$ $U = e^{i\chi^a t^a}$

$$\delta\psi = i\chi^a t^a \psi$$

$$D_\mu \psi = (2 - ig A_\mu^a t^a) \psi$$

Adjoint $F_{\mu\nu} = t^a F_{\mu\nu}^a$ $F_{\mu\nu} \rightarrow U F_{\mu\nu} U^\dagger$

$$\delta F_{\mu\nu} = i [\alpha^b t^b, t^a F_{\mu\nu}^a]$$

$$t^a \delta F_{\mu\nu}^a = -\alpha^b f^{bac} t^c F_{\mu\nu}^a$$

$$\delta F_{\mu\nu}^a = -f^{abc} \alpha^b F_{\mu\nu}^c$$

$$(t^b)_ac = -i \delta^{bac}$$

$$\delta F_{\mu\nu}^a = i \alpha^b (t^b_a)_{ac} F_{\mu\nu}^c$$

transformation law is the same $t^b_n \rightarrow t^b_a$

$$D_\mu F_{\rho\sigma}^a = (D_\mu \delta_{ac} - ig A_\mu^b (t^b_a)_{ac}) F_{\rho\sigma}^c$$

$$D_\mu \tilde{F}_{\rho\sigma}^a = D_\mu F_{\rho\sigma}^a + g f^{abc} A_\mu^b F_{\rho\sigma}^c$$

Yang-Mills field equations

$$D^\mu F_{\mu\nu}^a = -g j_\nu^a$$

$$\epsilon^{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma}^a = 0$$

$$j_\nu^a := \bar{\psi}^\dagger \gamma_\nu t^a \psi$$

(Branchi Identity)

DED: $D^\mu F_{\mu\nu} = e j_\nu$ $\epsilon^{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma} = 0$

$$\mathcal{L} = -\frac{1}{2} \text{Tr} [F_{\mu\nu} F^{\mu\nu}] + g \bar{\psi} i^\alpha \gamma_\mu \psi$$

$$= -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + g A_a^a j^{am}$$

$$\delta \mathcal{L} = -\frac{1}{2} F_{\mu\nu}^a \delta F_{\mu\nu}^a + g \delta A_a^a j^{am}$$

$$\begin{aligned} \delta F_{\mu\nu}^a &= 2 \delta A_\nu^a - \partial_\nu \delta A_\mu^a + g f^{abc} \delta A_\mu^b A_\nu^c + g f^{abc} A_\mu^b \delta A_\nu^c \\ &= \partial_\mu \delta A_\nu^a + g f^{abc} A_\mu^b \delta A_\nu^c - (\mu \leftrightarrow \nu) \end{aligned}$$

$$\delta \mathcal{L} = -F_{\mu\nu}^a (2 \delta A_\nu^a + g f^{abc} A_\mu^b \delta A_\nu^c) + g \delta A_\nu^a j^{av}$$

$$\delta S = \int d^4x \delta \mathcal{L} = \int d^4x -F_{\mu\nu}^a (2 \delta A_\nu^a + g f^{abc} A_\mu^b \delta A_\nu^c) + S \delta A_\nu^a j^{av}$$

↑
int. by parts

$$= \cancel{\int d^4x \delta A_\nu^a}$$

$$= \int d^4x \{ A_\nu^a [2 F_{\mu\nu}^a + g f^{abc} A_\mu^b F^{c\nu} + g j^{av}] \}$$

$$= \int d^4x \{ A_\nu^a [D_\mu F^{\mu\nu} + g j^{av}] \}$$

$$\Rightarrow \frac{\delta S}{\delta A_\nu^a} = 0 \Rightarrow \underline{D_\mu F^{\mu\nu} = -g j^\nu}$$

Quantization of Yang-Mills

Many Feynman rules for YM can be read off from lagrangian.

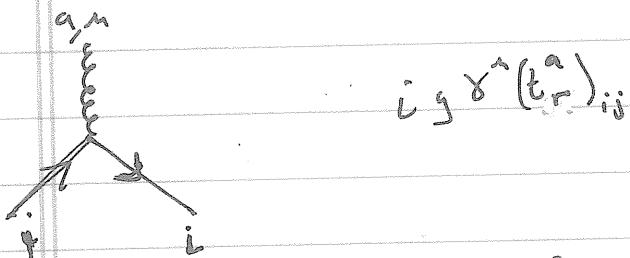
Interactions

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi} (i\gamma^\mu - g t^a A^a) \psi$$

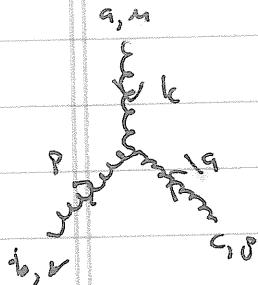
$$\mathcal{L}_0 = -\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + \bar{\psi} (i\gamma^\mu - m) \psi = \mathcal{L}$$

$$\begin{aligned} \mathcal{L}_{YM} &= \mathcal{L}_0 - \frac{1}{2} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) g f^{abc} A_\mu^b A_\nu^c - \frac{1}{4} g^2 f^{abc} A_\mu^b A_\nu^c f^{ade} A_\mu^d A_\nu^e \\ &\quad + g \bar{\psi} t^a \psi \end{aligned}$$

$$= \mathcal{L}_0 + g \bar{\psi} t^a \psi - g f^{abc} \partial_\mu A_\mu^a A_\nu^b A_\nu^c - \frac{g^2}{4} f^{abc} f^{ade} A_\mu^b A_\nu^c A_\mu^d A_\nu^e$$



$k_m \rightarrow -ik_m$ when k points into vertex



$$g f^{abc} \left[g^{uv} (k-p)^v + g^{uv} (l-q)^v + g^{uv} (q-k)^v \right]$$

the six terms come from six possible contractions

$$\begin{aligned}
 & \langle 0 | i \mathcal{L}_{\text{int}} | A^{a,m}(k) A^{b,v}(p) A^{c,j}(q) \rangle \\
 & = \langle 0 | -ig f^{abc} g_{jk} A^{a,i}_k A^{b,v}_j A^{c,j}_l | A^{a,m}(k) A^{b,v}(p) A^{c,j}(q) \rangle \\
 & = -ig f^{abc} -ik^v g_{jk} \\
 & = -g f^{abc} k^v g_{jk}
 \end{aligned}$$

Now consider six possible permutations of
 $\{a, m; k\}$ $\{b, v; p\}$ $\{c, j; q\}$

e.g.

$$3. \left\{ \begin{matrix} a \\ m \\ k \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} b \\ v \\ p \end{matrix} \right\} \quad -ig f^{bac} g_{vp} = ig f^{abc} g_{vp} \quad \text{etc.}$$

$$\begin{aligned}
 & \text{Diagram: } \begin{array}{ccc} a & b & c \\ m & v & r \\ k & p & s \end{array} \quad \text{contraction: } g_{m,v} g_{r,s} \\
 & = -ig^2 \{ f^{abc} f^{cde} (g_{m,p} g_{v,r} - g_{m,r} g_{v,p}) \\
 & \quad + f^{ace} f^{bde} (g_{m,v} g_{s,r} - g_{m,r} g_{v,s}) \\
 & \quad + f^{ade} f^{bce} (g_{m,v} g_{s,a} - g_{m,a} g_{v,s}) \}
 \end{aligned}$$

24 possible contractions, 4 are equivalent: $[A,B][C,D] = [B,A][D,C] = [D,C][B,A] = [C,D][A,B]$

This is it for interactions.

Fermion propagator: $\frac{i}{p} \frac{i}{p-m} \delta_{ij}$

For gauge boson, ghosts we need Faddeev-Popov procedure

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To derive gauge boson prop. (+ ghosts)

we use Faddeev-Popov procedure

Basic problem

Abelian & Non-Abelian G.T.

$$S = \int d^4x -\frac{1}{4} (2_\mu A_\nu - 2_\nu A_\mu) (2^\mu A^\nu - 2^\nu A^\mu)$$

$$= \int d^4x -\frac{1}{2} 2_\mu A_\nu (2^\mu A^\nu - 2^\nu A^\mu)$$

$$= \int d^4x \frac{1}{2} A_\mu 2_\nu 2^\mu 2^\nu A^\mu - A_\nu 2_\mu 2^\nu A^\mu$$

$$= \int d^4x \frac{1}{2} A_\nu [\square g^{\mu\nu} - 2^\mu 2^\nu] A_\mu$$

$$= \frac{1}{(2\pi)^4} \frac{1}{2} \tilde{A}_{\nu, k} [-g^{\mu\nu} k^2 + k^\mu k^\nu] \tilde{A}_{\mu, k}$$

($\tilde{A}_{\nu, k}$ is F.T. of $A_\nu(k)$)

now space gauge boson prop. is i times

inverse of $-g^{\mu\nu} k^2 + k^\mu k^\nu$. Problem: this is

not invertible. $(-g^{\mu\nu} k^2 + k^\mu k^\nu) k_\nu = -k^\mu k^2 + k^2 k^\mu = 0$

so $-g^{\mu\nu} k^2 + k^\mu k^\nu$ clearly has zero eigenvalue
and is not invertible.

Now suppose what we could impose Lorentz
gauge condition

$$2_m A^m = 0 \Rightarrow k_m \tilde{A}_m = 0$$

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$$\text{Then } S[A] = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{2} [\tilde{A}_{\nu,-k} [F^{\mu\nu} k^\nu] \tilde{A}_{\mu,k}]$$

$$\text{inverse} = i \frac{-g^{\mu\nu}}{k^2}$$

gauge boson propagator in Feyn. gauge

Actually we could have added a more general gauge fixing term. Consider

$$\begin{aligned} S_{\text{G.F.}}[A] &= -\frac{1}{2\beta} \int d^4 x (2\pi A^\mu)^2 \\ &= -\frac{1}{2\beta} \int \frac{d^4 k}{(2\pi)^4} \tilde{A}_{\mu,-k} k^\mu k^\nu \tilde{A}_{\nu,k} \end{aligned}$$

$$S[A] + S_{\text{G.F.}}[A] = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{2} \tilde{A}_{\nu,-k} \left[-g^{\mu\nu} k^\nu + \left(1 - \frac{1}{\beta}\right) k^\mu k^\nu \right] \tilde{A}_{\mu,k}$$

$$\begin{aligned} \text{To find inverse} & \quad \left(-g^{\mu\nu} k^\nu + \left(1 - \frac{1}{\beta}\right) k^\mu k^\nu \right) \left(A \delta_{\nu k} + B k_\nu k_\mu \frac{1}{k^2} \right) \\ &= -A S^\mu_\nu k^\nu + k^\mu k_\nu \left[A \left(1 - \frac{1}{\beta}\right) - B + B \left(1 - \frac{1}{\beta}\right) \right] \\ &= -A k^2 S^\mu_\nu + \frac{k^\mu k_\nu}{\beta} [A(\beta-1) - B] \\ &\equiv S^\mu_\nu \end{aligned}$$

$$A = -\frac{1}{k^2}, \quad B = (\beta-1)k = \frac{(1-\beta)}{k^2}$$

$$\Rightarrow \boxed{i \left(\frac{-g^{\mu\nu} + (1-\beta) k^\mu k^\nu / k^2}{k^2} \right) = \text{inverse}}$$

 $\beta = 0$ Landau gauge $\beta = 1$ Feyn. gauge

Useful check on actual AED calculation (of S-matrix elements) is 3 independence.

Problem: we started w/

$$\int dA e^{iS[A]}$$

and to get a sensible g.b. propagator we added by hand an extra term

$$\rightarrow \int dA e^{iS[A] + iS_{\text{g.b.}}[A']}$$

Defined: $A_\mu^R = A_\mu + \frac{i}{e} 2_R \Sigma$ (aED)

$$A_\mu^R = A_\mu^a + \frac{i}{g} 2_\mu \Sigma^a + f^{abc} A_\mu^b \Sigma^c \quad (\text{YM})$$

$$A_\mu^a = A_\mu^a + \frac{i}{g} D_\mu \Sigma^a$$

For YM this is infinitesimal of

$$A_\mu' = u A_\mu u^\dagger - \frac{i}{g} 2_\mu u u^\dagger \quad u = e^{iR^a t^a} = 1 + iR^a t^a + \dots$$

$$A_\mu'^a t^a = t^a A_\mu^a + i[R^a t^a, A_\mu^b t^b] + \frac{2_\mu R^a t^a}{g}$$

$$= t^a A_\mu^a - f^{abc} A_\mu^b \Sigma^c t^c - \frac{t^a 2_\mu R^a}{g}$$

$$A_\mu'^a = A_\mu^a + 2_\mu \Sigma^a + f^{abc} A_\mu^b \Sigma^c = A_\mu^a + \frac{i}{g} D_\mu \Sigma^a$$

Now for a gauge invariant action

$$S[A_m^R] = S[A_m]$$

$$\text{But } S_{GF}[A_m^R] = -\frac{i}{2S} \int d^4x \left(2^a A_m + \frac{1}{e} 2^a \partial_m \Sigma \right)^2$$

$$= S_{GF}[A_m + \frac{1}{e} 2^a \Sigma]^2 \quad \text{Q.E.D.}$$

$$= S_{GF}[A_m + \frac{1}{e} 2^a \Sigma] \quad \text{Yay.}$$

$$\neq S_{GF}[A_m]$$

So the gauge invariance of path integral
is no longer obvious when we add $S_{GF}[A_m]$

We can fix this problem by the
replacement

$$\int D[A_m] e^{iS[A_m]} \rightarrow \int D[A_m] D\Sigma D\omega e^{-\frac{i}{2S} \int d^4x \omega^2} e^{iS[A_m]} \Delta[A_m^R] S[G[A_m^R] - \omega]$$

where $\Delta[A_m^R]$ (Faddeev-Popov determinant)
is defined so that

$$\int D\Sigma \Delta[A_m^R] S(G[A_m^R] - \omega) = 1$$

for arbitrary ~~free~~ functions ω, A_m

Here $G[A_m] = \partial_m A^m$ is the gauge
fixing condition.

This is clearly equivalent to original functional integral if we do ω integral first.

$$\begin{aligned} \int D\tilde{A}_\mu e^{iS[\tilde{A}_\mu]} &= \int D\tilde{A}_\mu D\omega D\omega e^{-\frac{i}{2}\int d^4x \omega^2} e^{iS[\tilde{A}_\mu]} \delta(\tilde{A}_\mu^\alpha) \delta(\dots) \\ &= \int D\omega e^{-\frac{i}{2}\int d^4x \omega^2} \int D\tilde{A}_\mu e^{iS[\tilde{A}_\mu]} \\ &\propto \int D\tilde{A}_\mu e^{iS[\tilde{A}_\mu]} \end{aligned}$$

$\int D\omega e^{-\frac{i}{2}\int d^4x \omega^2}$ - irrelevant overall constant

However if we do ω integral first

$$\begin{aligned} \int D\tilde{A}_\mu e^{iS[\tilde{A}_\mu]} &= \int D\tilde{A}_\mu D\omega e^{iS[\tilde{A}_\mu]} e^{-\frac{i}{2}\int d^4x (G[\tilde{A}_\mu^2])^2} \Delta[\tilde{A}_\mu^2] \\ &= \int D\tilde{A}_\mu D\omega e^{iS[\tilde{A}_\mu] + iS_{\text{g.f.}}[\tilde{A}_\mu^2]} \Delta[\tilde{A}_\mu^2] \end{aligned}$$

~~measure of functional integral~~, measure of functional integral is gauge invariant.

$$\int D\tilde{A}_\mu = \int D\tilde{A}_\mu^R \quad \& \quad \text{use } S[\tilde{A}_\mu] = S[\tilde{A}_\mu^R]$$

$$\int D\tilde{A}_\mu e^{iS[\tilde{A}_\mu]} = \int DR \left[\int D\tilde{A}_\mu^R \Delta[\tilde{A}_\mu^R] \right] e^{iS[\tilde{A}_\mu^R] + iS_{\text{g.f.}}[\tilde{A}_\mu^R]}.$$

now A_μ^R is just a integration variable $A_\mu^R \rightarrow A_\mu$

$$= \int DR \left[\int D\tilde{A}_\mu \Delta[\tilde{A}_\mu] \right] e^{iS[\tilde{A}_\mu] + iS_{\text{g.f.}}[\tilde{A}_\mu]}$$

(36)

$\int dR \rightarrow$ just an overall constant we absorb this factor into normalization of path integral

so we can add $S_{\text{a.f.}}[A_m]$ to action provided we insert F.P. determinant, $\Delta[A_m]$ which is defined by:

$$\int dR \Delta[A_m^R] \delta[G[A_m^R] - \omega] = 1$$

$$\Delta[A_m^R] = \det \left(\frac{\delta G[A_m^R]}{\delta R} \right)$$

$$\boxed{\int dR \det \left(\frac{\delta G[A_m^R]}{\delta R} \right) \delta[G[A_m^R] - \omega] = 1}$$
(A)

recall for 1-D integral $\int dx \delta(f(x) - y) = \int \frac{df}{dx} \delta(f - y) = 1$

$$\text{so } \int dx \left| \frac{df}{dx} \right| \delta(f(x) - y) = \int df \delta(f - y) = 1$$

for many N variable integral

$$\int \prod dx^i \det \left| \frac{\partial f^i}{\partial x^j} \right| \delta \left(f^i(x^i) - s^i \right) = 1$$

(A) is a functional generalization of this formula

For QED (in Lorentz gauge)

$$L(A_\mu^R) = \partial^\mu A_\mu^R = \partial^\mu A_\mu + \frac{1}{e} \partial^\mu \partial_\mu R$$

$$\frac{\delta L(A_\mu^R)}{\delta R} = \frac{1}{e} \partial^2 \Delta[A_\mu] e \det\left(\frac{1}{e} \partial^2\right)$$

This is A_μ independent. Another irrelevant divergent constant. For YM,

$$S(A_\mu^R) = \partial^\mu A_\mu^R = \partial^\mu A_\mu + \frac{1}{g} \partial^\mu D_\mu R.$$

$$\Delta[A_\mu^R] = \det\left(\frac{\delta S(A_\mu^R)}{\delta R}\right) = \det\left(\frac{1}{g} \partial^\mu D_\mu\right)$$

In Perturbative QFT, we represent these determinant by functional integrals over Grassmann fields.

$$\det\left(\frac{1}{g} \partial_\mu D^\mu\right) = \int D\bar{c} Dc e^{i \int d^4x \bar{c} \left(-\frac{1}{g} \partial^\mu D_\mu\right) c}$$

$$c \rightarrow \sqrt{g} c$$

$$= \int D\bar{c} Dc e^{i \int d^4x \bar{c} \left(-\partial^\mu D_\mu\right) c}$$

$$\det(1/2 \partial^\mu) = \int D\bar{c} Dc e^{i \int d^4x \left(\partial^\mu \bar{c}^\alpha \partial_\mu c^\alpha + g \partial^\mu \bar{c}^\alpha g_{\mu\nu} c^\nu A_\mu^\beta c^\beta\right)}$$

(38)

Review of Grassmann variables.

I imagine 1...N fermionic d.o.f. Canonical commutation relations:

$$\{Q_a, P_b\} = i \delta_{ab} \quad \{Q_a, Q_b\} = \{P_a, P_b\} = 0$$

This is finite dimensional analog of what is obtained in aFT

$$L = \bar{\psi} (i\gamma^\mu - m) \psi \quad P_\psi = \frac{\delta L}{\delta \dot{\psi}} = i\bar{\psi} \gamma^0 = i\bar{\psi}^\dagger$$

$$\{\psi(x), \psi(y)\} = 0 \quad \{\psi(x), \psi^\dagger(y)\} = \delta(x-y)$$

Suppose we have a state $|q\rangle$ s.t $a_\alpha |q\rangle = q_\alpha |q\rangle$
 $a_\beta |q\rangle = q_\beta |q\rangle$

$$(a_\alpha a_\beta |q\rangle) = -Q_a Q_b |q\rangle \Rightarrow \underbrace{q_\alpha q_\beta + q_\beta q_\alpha}_{=0} = 0$$

→ this is why we must represent fermions by anticommuting numbers in path integral

$$(a_\alpha a_\beta |q\rangle) = Q_a Q_b |q\rangle = -Q_b Q_a |q\rangle = -q_\beta q_\alpha |q\rangle = q_\alpha q_\beta |q\rangle = q_\alpha q_\beta |a\rangle$$

of anticommuting numbers - Grassmann variables

(39)

Integration over Grassmann variables

$$\int d\eta f(\eta) = \int d\eta (f_0 + f_1 \eta) = f_1$$

$$\boxed{\int d\eta = 0 \quad \int d\eta \eta = 1}$$

$$\int d\eta (1, \eta) = \frac{d}{d\eta} (1, \eta) !$$

Definition preserves shift invariance $\int d\eta \eta = \int d\eta' (\eta' + a)$

$$\text{let } \eta = \eta' + a$$

Also this definition allows us to int. by parts

since:

$$\int d\eta \frac{d}{d\eta} f(\eta) = \int d\eta f_1 = 0.$$

Multiple Grassmann integrals

$$\int d\eta_1 d\eta_2 \dots d\eta_n \cdot (\eta_1 \eta_2 \dots \eta_n)$$

$$\int d\eta_1 (d\eta_2 \dots (d\eta_n \eta_n) \dots \eta_2) d\eta_1 = 1$$

$$d\eta_i d\eta_j = -d\eta_j d\eta_i$$

$$\eta_i \eta_j = -\eta_j \eta_i$$

~~Complexification~~ Commonly will use complex conjugate pairs

$$\eta = \frac{\eta_1 + i\eta_2}{\sqrt{2}} \quad \bar{\eta} = \frac{\eta_1 - i\eta_2}{\sqrt{2}}$$

$$\int d\bar{\eta} d\eta \eta \bar{\eta} = \underbrace{\int d\eta_1}_{\sqrt{2}} \underbrace{\int d\eta_2}_{\sqrt{2}} \underbrace{\frac{(\eta_1 + i\eta_2)(\eta_1 - i\eta_2)}{\sqrt{2}}}_{\sqrt{2}} = i^2 \int d\eta_1 d\eta_2 \eta_1 \eta_2 =$$

(40)

Gaussian integrals

$$\int d\bar{\eta} d\eta e^{-\lambda \bar{\eta} \eta} = \int d\bar{\eta} d\eta (\lambda - \bar{\eta} \eta)$$

$$\rightarrow \int d\bar{\eta} d\eta \eta \bar{\eta} = \lambda$$

For matrices $\int \prod_i d\bar{\eta}_i d\eta_i e^{-\bar{\eta}_i A_{ij} \eta_j}$

Assume A_{ij} is Hermitian $\eta_i \rightarrow u_{ij} \eta_j$

$$\text{s.t. } u^\dagger A u = \text{diag}(\lambda_i)$$

$$\rightarrow \int \prod_i d\bar{\eta}_i d\eta_i e^{-\bar{\eta}_i \lambda_i \eta_i} = \prod_i \lambda_i = \det A.$$

Compare w/

$$\int_{-\infty}^{\infty} dx_i e^{-x_i M_{ij} x_j} = \frac{(\pi)^{n/2}}{[\det M]^{1/2}}$$

Props One more interesting property $\int d\theta \theta = 1$ $\theta = \theta^i a^i$
 $d\theta = \frac{d\theta^i}{a^i}$

$$\therefore \int d\theta^i d\theta^i = 1$$

$$\Rightarrow \theta_i = M_{ij} \theta'_j \quad \prod_i d\theta_i = (\det M)^{-1} \prod_i d\theta'^i$$

as opposed to what we get for bosonic variables

$$x_i = M_{ij} x'_j \quad \prod_i dx_i = (\det M) \prod_i dx'_j$$

(4)

In Field theory, evaluate functional determinants using Grassmann path integral

$$\boxed{\text{Det}[\mathcal{D} + i\mathbf{m}] = \int [D\bar{\psi}] [D\psi] e^{i \int d^4x \bar{\psi} (i\mathcal{D} - m) \psi}} \quad (1)$$

What is $\text{Det}[\mathcal{D} + i\mathbf{m}]$

$$\text{Det}[\mathcal{D} + i\mathbf{m}] = \text{Det}[\mathcal{D}(\mathcal{D} + i\mathbf{m})^{-1}] = \text{Det}[-\mathcal{D} + i\mathbf{m}]$$

$$\Rightarrow \text{Det}[\mathcal{D} + i\mathbf{m}] = \sqrt{\text{Det}(\mathcal{D} + i\mathbf{m}) \text{Det}(-\mathcal{D} + i\mathbf{m})}$$

$$= \sqrt{\text{Det}(-D^2 - m^2)}$$

works for $D_m = \partial_m$
or $D_m = \partial_m + i\omega_m$

using $\text{Det} M = \exp T_r[\ln M]$

$$\text{Det}[\mathcal{D} + i\mathbf{m}] = e^{\frac{1}{2} T_r [\ln(-D^2 - m^2)]}$$

$$= e^{\frac{Vd}{2} \int \frac{d^4k}{(2\pi)^4} \ln(k^2 - m^2)} = \infty ! \quad \begin{matrix} \text{VT-dimension} \\ \text{or space-time} \end{matrix}$$

We can at least test whether (1) gives expected A_m dependence (for fixed $A_m(x)$)

$$\text{Det}[\mathcal{D} + i\mathbf{m}] = \text{Det}[-\mathcal{D} + ieA + i\mathbf{m}]$$

$$= \text{Det}[(\mathcal{D} + i\mathbf{m}) \left(1 - \frac{i}{i\mathcal{D} - m} (-ieA) \right)]$$

(414)

Attempt to interpret $\text{Det}(\not{p} + \text{im})$

$$\text{Tr}[\ln(-\not{p}^2 - m^2)] = V_d \int \frac{d^d k}{(2\pi)^d} \ln(k^2 - m^2 + i\epsilon) \times 4 \xleftarrow{\text{Tr}_q[1] \text{ in Dirac matrix}}$$

$$- \frac{d}{dm^2} \int \frac{d^d k}{(2\pi)^d} \ln(k^2 - m^2 + i\epsilon) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2 + i\epsilon}$$

$$= \int \frac{dk_x}{2\pi} \frac{d^{d-1} k}{(2\pi)^{d-1}} \frac{1}{k_x^2 + \omega_k^2 + i\epsilon}$$

$$= \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \frac{-i}{2\omega_k}$$

$$= -i \frac{d}{dm^2} \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \omega_k$$

$$\omega_k = \sqrt{k^2 + \omega_k^2}$$

$$k_x = \omega + ik$$

$$\omega_k - i\epsilon$$

$$\Rightarrow \boxed{\int \frac{d^d k}{(2\pi)^d} \ln(k^2 - m^2 + i\epsilon) = i \int \frac{d^{d-1} k}{(2\pi)^{d-1}} (\omega_k + \text{const.})}$$

$$\Rightarrow \text{Det}(\not{p} + \text{im}) = \exp \left[i V_d 4 \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \frac{\omega_k}{2} \right]$$

↑ vacuum energy

density (-ve for fermions)

$$\langle 0 | e^{-iH\tau} | 0 \rangle = e^{-iE\tau} \\ = e^{-i\beta_0 V T}$$

$$E = \beta_0 V$$

[energy density]

$$\text{Note } \int [D\bar{\psi}] [D\psi] e^{-\lambda \int \bar{\psi} \hat{\partial}^4 \psi} \Rightarrow \int D\bar{\psi} D\psi e^{-\int \bar{\psi} \hat{\partial}^4 \psi} \quad (\lambda \rightarrow \frac{1}{\lambda})$$

so numerical coefficient multiplying λ is irrelevant

(42)

$$\frac{\text{Det} [\gamma + im]}{\text{Det} (\gamma + im)} = \text{Det} \left[1 - \frac{i}{im} (-ieA) \right]$$

$$= \exp \text{Tr} \ln \left(1 - \frac{i}{im} (-ieA) \right)$$

$$= \exp \left[- \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \left[\frac{i}{im} (-ieA) \right]^n \right]$$

This is exactly what path integral in (1)
yields:

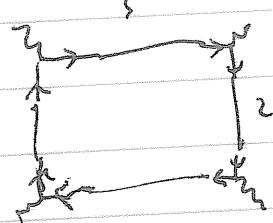
$$\int D\bar{\psi} D\psi e^{i \int d^4x \sqrt{i\theta-m} \bar{\psi} i \not{D} \psi}$$

$$= \exp \left[\text{Diagram } A + \text{Diagram } A + t \text{Diagram } A + t^2 \text{Diagram } A + \dots \right]$$

-1 - fermion loop

$\frac{1}{n}$ - symmetry factor

symmetry factor for four A_μ



$$\frac{1}{4!} \cdot 3 \cdot 2 \cdot 1 = \frac{1}{4}$$

ways to choose 1st contraction
↓ ways " " 2nd

(43)

Complete YM Lagrangian after gauge fixing

$$L = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2g} (\partial^\mu A_\mu^a)^2 + g \bar{c}^a \partial_\mu c^a + \bar{\psi} (i\gamma^\mu - m) \psi$$

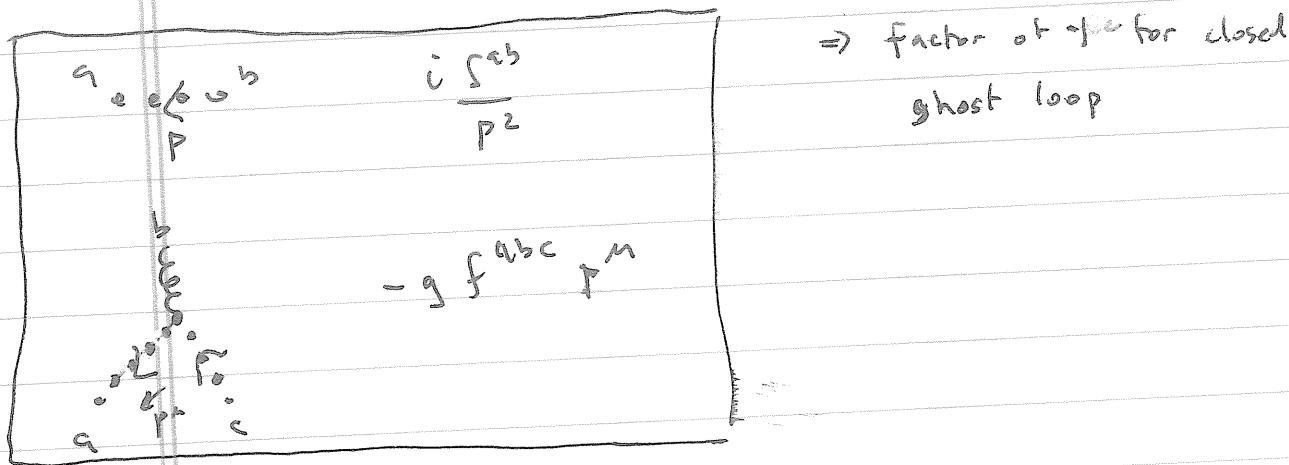
$$L' = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2g} (\partial^\mu A_\mu^a)^2 + g \bar{c}^a \partial_\mu c^a + g f^{abc} g^{mn} \bar{c}^a A_\mu^b c^m + \bar{\psi} (i\gamma^\mu - m) \psi$$

Additional Feynman rules.

G.B propagator

$$\frac{a_\mu^a}{k} \text{ vertex } \frac{i\delta^{ab}}{k^2} \left(-g^{\mu\nu} + (1-s) \frac{k^\mu k^\nu}{k^2} \right)$$

ghosts: c^a : spin-one, anti commuting (-)



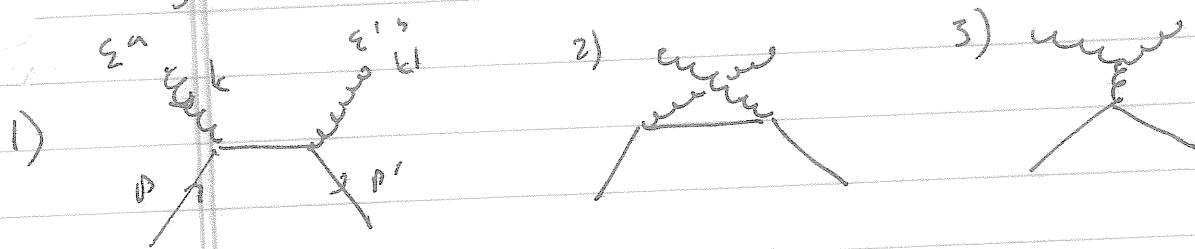
$$+ g f^{abc} \bar{c}^a A^b c^c$$

↑ annihilates incoming

creates outgoing

$i p^a$ for outgoing momentum

Some checks on our rules: $g_2 \rightarrow g_2$



Graphs 1) + 2) are QED-like, 3) unique

to non-abelian gauge theory. Expect result similar to what we found in QED

$$\text{if } \overset{n}{\epsilon_{\mu\nu}} M_n \rightarrow \overset{n}{k^\mu} M_n = 0$$

- longitudinally polarized gluon should decouple from S-matrix

- Lorentz invariance

- For this identity to hold all other particles must be on-shell (including gluons)

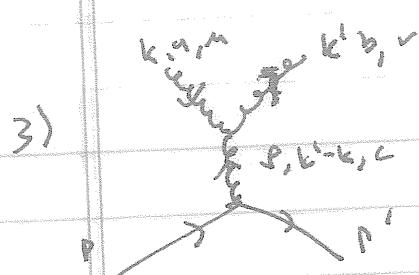
1st 1) + 2) as in QED

$$1) + 2) : -ig^2 \bar{u}(p') \left[\not{p}' \not{k}' \not{t}^b + \not{p}' \not{t}^a \not{k}' \not{t}^b \right] u(p)$$

$\not{q} \rightarrow \not{k}' + \text{use Dirac eqn.}$

$$= -ig^2 \bar{u}(p') \not{t}^b [\not{t}^b, \not{t}^a] u(p)$$

$$= -g^2 \bar{u}(p') \not{t}^b \not{t}^a \not{t}^b u(p) = 0$$



Feynman gauge for gauge boson propagator.

$$= \bar{u}(p') i g t^c \gamma^\mu u(p) - i \frac{g f^{abc}}{(k-k')^2} \left\{ (k+k')_\mu S^{\mu\nu} + (-2k'+k)_\mu \delta_\nu^\mu \right. \\ \left. + (k'-2k)_\nu \delta_\mu^\nu \right\} \bar{\varepsilon}_\mu^{(a)} \varepsilon_\nu^{(b)}$$

$$\text{using } k_\mu \varepsilon^\mu(a) = 0 \quad k'_\mu \varepsilon^\mu(b) = 0$$

$$= g^2 f^{abc} \bar{u}(p') t^c \gamma^\mu u(p) \frac{1}{(k-k')^2} \left[(k+k')_\mu \varepsilon \cdot \varepsilon' - 2k' \cdot \varepsilon \varepsilon'_\mu - 2k \cdot \varepsilon' \varepsilon_\mu \right]$$

Now if we replace $\varepsilon \rightarrow k_g$ $k^2 = k'^2 = 0$

$$= g^2 f^{abc} \bar{u}(p') t^c \gamma^\mu u(p) \frac{1}{-2k \cdot k'} \left[(k+k')_\mu k \cdot \varepsilon' - 2k \cdot k \varepsilon'_\mu - 2k \cdot \varepsilon' k_\mu \right]$$

$$= g^2 f^{abc} \bar{u}(p') t^c \gamma^\mu u(p) \frac{1}{-2k \cdot k'} \left[(-k+k')_\mu k \cdot \varepsilon' - 2k \cdot k \varepsilon'_\mu \right]$$

$$p+k = p+k'$$

$$p-p' = -k+k'$$

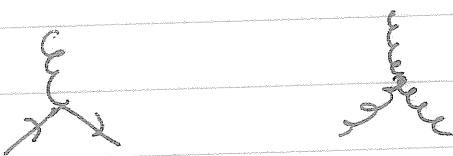
$$\text{now use } \bar{u}(p') \gamma^\mu u(p) (-k+k')_\mu \\ = \bar{u}(p') \gamma^\mu u(p) (p-p')_\mu \\ = \bar{u}(p') (p-p') u(p) \\ = \bar{u}(p') (u-u) u(p) = 0$$

$$3) \stackrel{k^2 = k'^2}{=} g^2 f^{abc} \bar{u}(p') t^c \gamma^\mu u(p) = -[(1+2)]$$

$$k^2 \left(m_1^{\mu} + m_2^{\mu} + m_3^{\mu} \right) = 0$$

(46)

Non-trivial test that are Feynman Rules
for

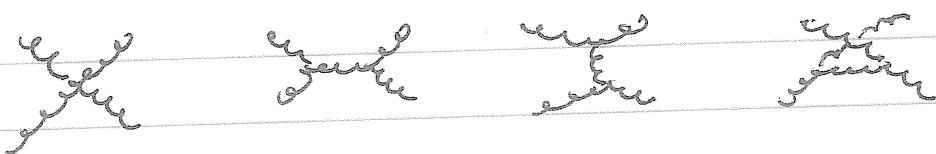


give sensible tree-level amplitudes - even with
on-shell, transverse gluons. ~~even~~

Note gauge invariance requires coupling
constants be the same in both F.R.
i.e.

$$g_{\bar{q}qA} = g_{AAA} = g$$

likewise considering similar cancellations
in



$$\text{Yield } S_{AAAA} = 0$$

Ghosts: Necessary for unitarity: Consider
one-loop diagrams for $q\bar{q} \rightarrow g g$

$$2 \text{ diagrams} + 2 \text{ diagrams} = \left[\text{tree level diagrams} \times \text{phase space} \right]$$

this fails for gluon self energy, need ghosts to fix this

