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Chiral symmetry and Anomalies

2-D QED

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \psi - \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

$$\{\gamma^m, \gamma^n\} = 2g^{mn} \quad m, n = 0, 1$$

Gamma matrices are 2-D

$$\gamma^0 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} & i \\ i & \end{pmatrix}$$

$$(\gamma^0)^2 = \mathbb{1} \quad (\gamma^1)^2 = -\mathbb{1}$$

Γ_5 4-D	$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$	$\gamma^5 = \gamma^5$
$\gamma^5 = \gamma^0\gamma^1$ 2-D		$\gamma^5 = \gamma^5$

$$\gamma_5 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \begin{pmatrix} & i \\ i & \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\Psi = \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} \quad \Psi_R = \begin{pmatrix} \psi_R \\ 0 \end{pmatrix} = \begin{pmatrix} 1+\gamma_5 \\ 2 \end{pmatrix} \Psi$$

$$\gamma_5 \Psi_R = \Psi_R \quad \gamma_5 \Psi_L = -\Psi_L$$

just like 4-D

(6)

$$D = \begin{pmatrix} -i(\partial_0 + \partial_1) \\ i(\partial_0 + \partial_1) \end{pmatrix}$$

$$\bar{\psi} i\cancel{D} \psi = \bar{\psi}^+ r^0 i\cancel{D} \psi$$

$$= \bar{\psi}^+ \begin{pmatrix} -i \\ i \end{pmatrix} \begin{pmatrix} \partial_0 - \partial_1 \\ -(\partial_0 + \partial_1) \end{pmatrix} \psi$$

$$= \bar{\psi}^+ \begin{pmatrix} i(\partial_0 + \partial_1) \\ i(\partial_0 - \partial_1) \end{pmatrix} \psi$$

$$= \bar{\psi}_R^+ i(\partial_0 + \partial_1) \psi_R + \bar{\psi}_L^+ i(\partial_0 - \partial_1) \psi_L$$

$$(\partial_0 \pm \partial_1) \psi_i = 0 \quad \psi = e^{i k_i (t \mp z)}$$

ψ_R - right movers

ψ_L - left movers

We expect currents for left/right

movers to be conserved when

there is no mass term for fermions.

$$j_R^\mu = \bar{\psi}_R \gamma^\mu \psi_R \quad j_L^\mu = \bar{\psi}_L \gamma^\mu \psi_L$$

$$\partial_\mu j_R^\mu = \partial_\mu j_L^\mu = 0 \quad \text{Classically}$$

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Vector current $j^m = \bar{\psi} \gamma^m \psi = \bar{\psi}_R \gamma^m \psi_R + \bar{\psi}_L \gamma^m \psi_L$
 axial vector $j_5^m = \bar{\psi} \gamma^m \psi = \bar{\psi}_R \gamma^m \psi_R - \bar{\psi}_L \gamma^m \psi_L$

Peculiarity of 2-D $\gamma^\mu \gamma_5 = -(\epsilon^{\mu\nu}) \gamma_5$

$$\epsilon^{01} = 1 = -\epsilon^{10} \quad \gamma^0 \gamma_5 = \gamma^0 \gamma^0 \gamma^1 = \gamma^1 = -\epsilon^{01} \gamma_1$$

$$\gamma^1 \gamma_5 = \gamma^1 \gamma^0 \gamma^1 = \gamma^0 = -\epsilon^{10} \gamma_0$$

So $j_5^m = -\epsilon^{\mu\nu} j_\nu$ ← not true in 4D

allows us to relate vector current to axial current. ✓

Vacuum polarization ~~vacuum~~

we have already evaluated this in D dimensions:

$$i\Pi^{\mu\nu}(q) = m \Omega_{\mu\nu} = -i (q^2 g^{\mu\nu} - q^\mu q^\nu) \frac{2e^2}{(4\pi)^2 r^2} \int dx \left[q^2 x (1-x) \right] \frac{\Gamma(D/2)}{\Gamma(D/2 - 2)} x(1-x) \text{Tr}[J]$$

This ' is trace over Dirac matrices, $\text{Tr}[1] = 4 \quad (D=4)$
 $\text{Tr}[1] = 2 \quad (D=2)$

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$$\frac{ze^2}{(4\pi)^{D/2}} \Gamma(1) \int_0^1 dx \times (1-x) [q^2(x(1-x))]^{\frac{D}{2}-2} \Gamma(2-\frac{D}{2})$$

$$\frac{e^2}{\pi} \frac{1}{-q^2} \int_0^1 dx \cdot \Gamma(1) =$$

$$\text{so } i\tilde{\Pi}^{\mu\nu}(q^2) = i \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \frac{e^2}{\pi}$$

Since this $\propto g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}$

rather than $\propto q^2 g^{\mu\nu} - q^\mu q^\nu$

this correction acts like a mass term
for the photon:

$$m_p^2 = \frac{e^2}{\pi}$$

(This is an exact result. Schwinger (1962))

Dimensions of fields: $\int_A = \int d^D x \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$$\Rightarrow [F_{\mu\nu}] = +1 \quad [A_\mu] = 1 \quad \Rightarrow [A_\mu] = 0$$

$$D_m = \partial_m + ie A_m$$

$$\Rightarrow [e] = 1$$

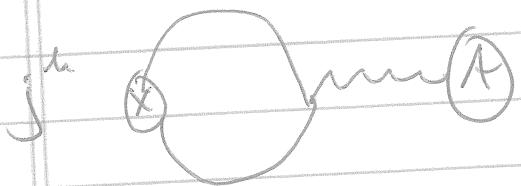
$$[\partial_m] = 1$$

Coupling constant has dimension 1!
One way to see why $m e^2$, (there
is no other dimensionful parameter in the
theory.)

Now let's consider expectation value of current

$$\int d^2x e^{i\vec{q} \cdot \vec{x}} \langle j^m(x) \rangle = \langle j^m(q) \rangle$$

in presence of a background field.



This is $\frac{i}{e} (i\pi^{\mu\nu}(q) A_\nu(q))$

$$\langle j^m(q) \rangle = - \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \frac{e}{\pi} A_\nu$$

The Ward identity is satisfied:

$$q_m \langle j^m \rangle = 0$$

But what about $\langle j_5^m(q) \rangle = -\epsilon^{\mu\nu} \langle j_\nu(q) \rangle$

$$q_m \langle j_5^m(q) \rangle = q_m \epsilon^{\mu\nu} \frac{e}{\pi} \left(A_\nu(q) - \frac{q_\nu q^2}{q^2} A_\lambda(q) \right)$$

$$(Dg) q_m \langle j_5^m(q) \rangle = \frac{e}{\pi} \epsilon^{\mu\nu} q_m A_\nu(q)$$

F.T. $\boxed{D_m j_5^m(x) = \frac{e}{\pi} \epsilon^{\mu\nu} F_{\mu\nu}} \neq 0$

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conservation of ~~current~~^{axial} current is violated by quantum effects.

There are many, many ways to compute anomalies. One method is point splitting

$$j_5^m(x) = \lim_{\delta \rightarrow 0} \bar{\psi}(x - \frac{\epsilon}{2}) e^{-i\int_x^{x + \frac{\epsilon}{2}} dz A(z)} \sqrt{\psi(x + \frac{\epsilon}{2})}$$

compute $\partial_m j_5^m$ find see Peskin for details

$$\lim_{\delta \rightarrow 0} \bar{\psi}(x + \frac{\epsilon}{2}) - i \epsilon^m \epsilon^\nu F_{\mu\nu} \delta^5 \psi(x + \frac{\epsilon}{2})$$

looks like it vanishes, but $\cancel{\psi} \cancel{u} \sim \frac{1}{\epsilon}$

and a finite piece remains

Global aspects

note $\epsilon^{\mu\nu} F_{\mu\nu} = 2 \epsilon^{\mu\nu} \partial_\mu (\epsilon^{\rho\sigma} A_\nu)$

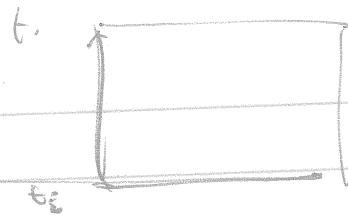
r.h.s. is a total derivative. (check)

Global charge associated w/ j_5^m is

$$N_R - N_L = \int dx j_5^m = \int dx \bar{\psi}_R \gamma^5 u_R - \bar{\psi}_L \gamma^5 u_L$$

$$\partial_m j_5^m = 0 \Rightarrow \frac{d}{dt} (N_R - N_L) = 0$$

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$$\int d^nx \partial_n j_5^n =$$

$$\int_{t_i}^{t_f} dt \int dx (j_0 \dot{j}_5 + j_1 \dot{j}_5')$$

surface integral vanishes
if $j_5'(x)$ is confined
to a finite volume of
space.

$$= \left[\int dx j_5^0 \right]_{t_i} - \left[\int dx j_5^0 \right]_{t_i}$$

$\Rightarrow \Delta(N_0 - N_1)$. So if we integrate the
anomaly equation

$$\int d^3x \partial_n j_5^n = e \underbrace{\int d^3x \epsilon^{mn} F_{mr}}_{= \Delta(N_0 - N_1)}$$

How does the right hand side know to
be an integer? It is a topological
invariant.

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$$A^I(x, t) = A^I f(t)$$

$f(t)$ adiabatically changing with time

$$1 - \int dx \frac{i\psi_R^\dagger(\partial_1 - ieA_1)}{\hbar} \psi_R + i\psi_R^\dagger(\partial_1 - ieA_1)$$

$$H = \int dx \psi_R^\dagger (-i\partial_1 - eA_1) \psi_R + \psi_R^\dagger (i\partial_1 - ieA_1) \psi_R$$

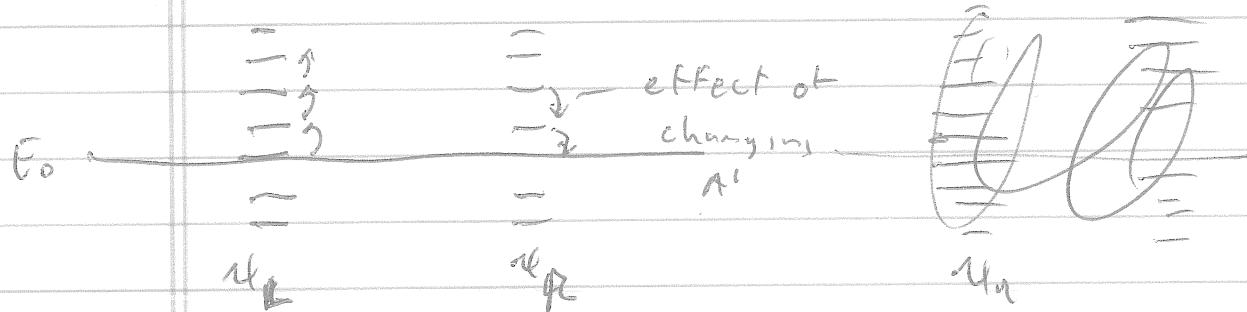
if $f(t)$ is ^{changing} adiabatic we can treat A as constant

$$E_n = k_n - eA^I \quad k_n = \frac{2\pi n}{L}$$

$$E_L = -(k_n - eA^I)$$

clearly $A^I \sim A^I + \frac{2\pi}{eL}$ are

gauge equivalent since they yield spectrum



now let $f(-\infty) = 0 \quad f(+\infty) = 1$

$$A^I = \frac{2\pi}{eL}$$

(a)

$$\begin{aligned}
 N_R - N_L &= \int d^x \frac{e}{\hbar} e^{i k \cdot x} F_{\mu\nu} \\
 &= \int dt \int d^x \frac{e}{\hbar} \partial_\mu A_\nu - \partial_\nu A_\mu \\
 &= \frac{e}{\hbar} L \cdot A_1(t=\infty) - A_1(t=-\infty) \\
 &= \frac{e}{\hbar} L \Delta A_1 \\
 &= \frac{e}{\hbar} L^{-1} A A' \\
 &= \underline{-2}
 \end{aligned}$$

One right handed fermion disappeared into vacuum, one left handed fermion was created. So everything worked out

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Anomaly in 4-D

$$\mathcal{L} = \bar{\psi}_L i\cancel{D} \psi_L + \bar{\psi}_R i\cancel{D} \psi_R - m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)$$

classically axial transformation is

$$q_L = e^{i\theta} q_L \quad q_R = e^{-i\theta} q_R$$

and Noether current associated w/
this symmetry is

$$\hat{j}_\mu^5 = \bar{\psi} \gamma_\mu \gamma_5 \psi \quad \partial^\mu j_\mu^5 = 0 \quad (\text{for } m=0)$$

Because of the mass term this should
not be conserved.

$$\begin{aligned} \delta \mathcal{L} &= i m \gamma_0 (\bar{\psi}_L \gamma_R - \bar{\psi}_R \gamma_L) = 2 i m \theta \bar{\psi} \gamma_5 \psi \\ &= \theta \partial_\mu j_\mu^5 \end{aligned}$$

$$\Rightarrow \boxed{\partial_\mu j_\mu^5 = 2 i m \bar{\psi} \gamma_5 \psi} \leftarrow \text{classical equation}$$

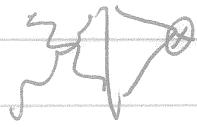
But quantum mechanically there is another
contribution when the fermion is coupled
to a gauge field.

Evaluate one loop Feynman diagrams

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$$A_{\mu}^{(k_1)} \text{ and } A_{\nu}^{(k_2)}$$

δL_{class}



$$- i2am \int \frac{d^4e}{(2\pi)^4} T_{\mu\nu} \frac{i}{k+k_1-m} (\text{field}) \frac{i}{k-m} (\text{field}) \frac{i}{k+k_2-m}$$

+ ($k_1 \leftrightarrow k_2, \mu \leftrightarrow \nu$)

$$= \alpha \frac{e^2}{(2\pi)^2} \epsilon_{\mu\nu\lambda\beta} k_1^\mu k_2^\nu \epsilon_1^\lambda \epsilon_2^\beta$$

$$\delta L' = -\alpha \frac{1}{16\pi^2} \epsilon^{\mu\nu\lambda\beta} F_{\mu\nu} F_{\lambda\beta} = -\frac{\alpha}{16\pi^2} \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\beta} F_{\lambda\beta}$$

so correct equation is

$$[2im \bar{\psi} \gamma_5 \psi - \frac{e^2}{8\pi^2} F^{\mu\nu} \tilde{F}_{\mu\nu}]$$

$\frac{1}{2}$
anomaly term

2nd term survives in $m \rightarrow 0$ limit
check this by carefully evaluations

$$\text{F.T.} \langle 0 | T \partial_{\mu} \bar{\psi}(x) \bar{\psi}(y) \bar{\psi}(z) | 0 \rangle$$

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$$C_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$$= 4\epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma$$

$$= 4 \partial_\mu [\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma]$$

so anomaly term is again a total derivative. This can also be shown for non-abelian field strength. Also has topological significance

$\frac{g^2}{16\pi^2} (N_x \text{Tr} [C_{\mu\nu} G^{\mu\nu}])$ is an integer.

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Anomalies in 4-D

generalization of our 2-D result for the anomaly is

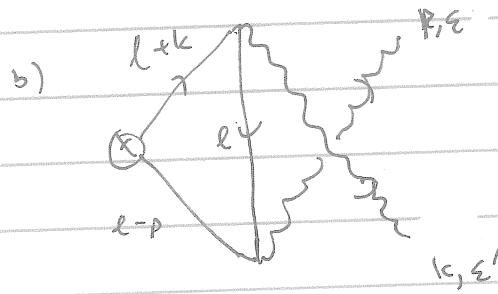
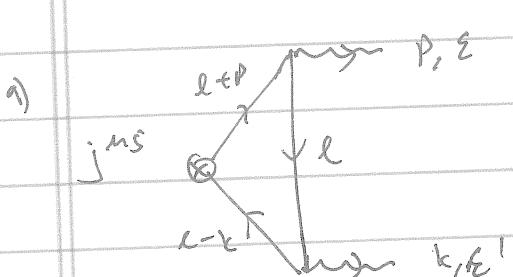
$$2\mu j^{MS} = -\frac{e^2}{16\pi^2} \epsilon^{\lambda\mu\nu} F_{\lambda\mu} F_{\nu\rho}$$

We can establish this by evaluating

$$\int d^4x e^{-i\varphi \cdot x} \langle p, k | j^{MS}(x) | 0 \rangle$$

↑
two photon state

Triangle diagrams.



a) $= (-i\epsilon)^2 \left[\frac{d^4 l}{(2\pi)^4} \text{tr} \left[\gamma^\mu \gamma_5 \frac{i(l-k)}{(l-k)^2} \not{l} \not{k} \not{l} \not{p} \right] \right]$

b) $= -i\epsilon^2 \left[\frac{d^4 l}{(2\pi)^4} \text{tr} \left[\gamma^\mu \gamma_5 \frac{(l-k)}{(l-k)^2} \not{l} \not{k} \not{(l+p)} \not{(l+p)} \right] \right]$

(2)

$$a) = (-1)^2 \left(\frac{d^4 e}{(2\pi)^4} \text{Tr} \left[\gamma^\mu r_5 \frac{i}{x-k} \not{e}' \frac{i}{x} \not{e} \frac{i}{x+k} \right] \right)$$

$$= -i e^2 \left(\frac{d^4 e}{(2\pi)^4} \text{Tr} \left[\gamma^\mu r_5 \frac{1}{x-k} \not{e}' \frac{1}{k} \not{e} \frac{1}{x+k} \right] \right)$$

~~relabel~~ b) = a) ($\rho \leftrightarrow k, \varepsilon \leftrightarrow \varepsilon'$)

To learn about $iq_m^{(a)}$

$$\left\langle d^4 x e^{-iq \cdot x} \langle p, k | \bar{u}_n j^{n5}(x) | 0 \rangle \right\rangle = iq_m \left\langle d^4 x e^{-iq \cdot x} \langle p, k | j^{n5}(x) | 0 \rangle \right\rangle$$

To bear evaluate $iq_m(a)$

$$\begin{aligned} g_m \gamma^\mu r_5 &= q r_5 = (\not{x} + \not{k}) r_5 \\ &= (\not{x} + \not{p} - (\not{x} - \not{k})) r_5 \\ &= (\not{x} - \not{p}) r_5 + r_5 (\not{x} - \not{k}) \end{aligned}$$

$$iq_m(a) = e^2 \left(\frac{d^4 e}{(2\pi)^4} \text{Tr} \left[r_5 \frac{1}{x-k} \not{e}' \frac{1}{k} \not{e} + r_5 \not{e}' \frac{1}{k} \not{e} \frac{1}{x+k} \right] \right)$$

$$iq_m(b) = e^2 \left(\frac{d^4 e}{(2\pi)^4} \text{Tr} \left[r_5 \frac{1}{x-k} \not{e} \frac{1}{k} \not{e}' + r_5 \not{e} \frac{1}{k} \not{e}' \frac{1}{x+k} \right] \right)$$

Naive inspection of this expression would indicate

$$iq_m(a) + (b) = 0$$

(3)

Consider underlined terms.

$$\left[\frac{d^4 c}{(2\pi)^n} \text{tr} \left| \underline{\gamma_5} \underline{\not{e}} \not{x} \not{\not{e}}' \not{\not{x}}' \right| - \underline{\gamma_5} \not{e} \underline{\not{x}} \not{\not{e}}' \not{\not{x}}' \right]$$

$\not{e} \rightarrow \not{e} + \not{x}$ in first term

$$\begin{aligned} & \text{tr} \left[\underline{\gamma_5} \underline{\not{e}} \not{\not{e}}' \not{\not{x}}' \right] + \text{tr} \left[\underline{\gamma_5} \not{e} \underline{\not{x}} \not{\not{e}}' \not{\not{x}}' \right] \\ &= - \text{tr} \left[\underline{\gamma_5} \not{e} \underline{\not{x}} \not{\not{e}}' \not{\not{x}}' \right] + \text{tr} \left[\underline{\gamma_5} \not{e} \underline{\not{x}} \not{\not{e}}' \not{\not{x}}' \right] \end{aligned}$$

= 0.

Problem: $\int d^4 x f(x) = f d^4 x f(x)$

when the integral is convergent! However, these integrals are divergent and one has to be more careful. For these particular integrals any regulator yields some extra finite term which does not cancel.

In dim. reg. problem is associated w/ proper definition of γ_5 .

(4)

$$4 \cdot D: \quad \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$$D=4-2G/ \quad \{ \gamma_5, \gamma^m \} = 0 \quad \text{if} \quad m=0, 1, 2, 3 \\ \text{but} \quad [\gamma_5, \gamma^m] = \gamma^m \quad m \neq 0, 1, 2, 3$$

The remaining "extra" dimensions commute w/ γ_5

(This is 'tHooft-Veltman definition. There are others.) So we will decompose

$$\ell = \ell_{\parallel} + \ell_{\perp}$$

$\underbrace{\phantom{\ell_{\parallel}}}_{\text{4 dimensions}}$ $\overbrace{\ell_{\perp}}^{\text{extra dimensions}}$

$$\text{Earlier we used } g \gamma_5 = (\ell + p) \gamma_5 + \gamma_5 (\ell - p)$$

This only works if $\ell \gamma_5 + \gamma_5 \ell = \{ \gamma_5, \ell \} = 0$
but in d-dimensions

$$\ell \gamma_5 + \gamma_5 \ell = 2 \gamma_{\perp} \gamma_5$$

$$\text{so } g \gamma_5 = (\ell + p) \gamma_5 + \gamma_5 (\ell - p) - 2 \gamma_5 \gamma_{\perp}$$

so in addition to terms that cancel we obtain:

$$g_a(a) := -2e^2 \int \frac{d^D k}{(2\pi)^D} \text{tr} \left[\gamma^5 \gamma_{\perp} \frac{1}{\ell - k} \gamma' \frac{1}{\ell} \gamma' \frac{1}{\ell + p} \right]$$

$$-2e^2 \int_{(2\pi)^0}^{\infty} \frac{dx}{x^2} \left[\frac{\gamma_5 \kappa_L (k-x) \not{k} \not{x} \not{(k+p)}}{(k-x)^2 x^7 (k+p)^2} \right]$$

$$\frac{1}{x^2 (k-x)^2 (k+p)^2} = \frac{1}{k^2 (k^2 - 2xk - x^2) (k^2 + 2xp - p^2)} = \int_0^1 dx \int_0^{1-x} dy \frac{1}{[k^2 - 2xk - x^2 + 2y(p-k)]^3}$$

$$k \rightarrow k + xk - xp = k + 2xp \quad \Rightarrow \int_0^1 dx \int_0^{1-x} dy \frac{1}{[k^2 + 2xp - x^2]^3}$$

$$= \int_0^1 dx \int_0^{1-x} dy \frac{1}{(k^2 + x^2)^3}$$

Numerator

after the shift: $\kappa_1 \rightarrow \kappa_1' \quad (\not{k}_2 \not{\kappa}_2 = 0)$

$$-4e^2 \int_0^1 dx \int_0^{1-x} dy \int_{(2\pi)^0}^{\infty} \frac{dx'}{(2\pi)^0} \frac{\gamma_5 \kappa_L [k - (1-x)\kappa - xp]}{(k^2 + x^2)^3} \not{k}' (\not{k} + x\not{k} - \not{p}) \not{k} (\not{k} + x\not{k} + (-x)\not{p})$$

Now we have to use symmetric integration.

$$\int_{(2\pi)^0}^{\infty} \frac{dx'}{(2\pi)^0} \frac{\kappa_L \not{k}}{(k^2 - 1)^3} = \int_{(2\pi)^0}^{\infty} \frac{\kappa_L (k_L + \not{k}_{\perp})}{(k^2 - 1)^3}$$

$$\kappa_L \not{k}_L = \frac{d^2 k}{d\kappa^2} = \frac{(\kappa-4)}{\kappa} k^2 = \frac{(-2\epsilon)}{4-2\epsilon} k^2$$

aha! $(\kappa-4)$ will cancel against $\frac{1}{\epsilon}$ pole

from the Feynman integral yielding a finite result.

(6)

terms w/ 4 ϵ 's will not contribute

$$\begin{aligned} & \text{tr} [Y_5 \gamma_L \not{x}' \not{\epsilon} \not{\epsilon}] \\ & \propto \text{tr} [\not{e}_L^2 \not{e}^2 Y_5 \gamma_5 \not{\epsilon}' \not{\delta}^m \not{\epsilon}] = \\ & \propto \text{tr} [\not{e}_L^2 \not{e}^2 Y_5 \not{\epsilon}' \not{\epsilon}] = 0. \end{aligned}$$

so we get

$$\begin{aligned} & \frac{4\epsilon^2 (4-D)}{D} \int_0^1 dx \int_0^{1-x} dy \int \frac{\not{e}_L}{(2\pi)^D} \frac{\not{x}^2}{(\not{e}' + xy\not{q}^2)} \\ & \times \left[\text{tr} [Y_5] - (1-x) \not{x} - \not{y} \not{p} \right] \not{\epsilon}' (\not{x} \not{k} - \not{y} \not{p}) \not{\epsilon} \left[\text{tr} [Y_5 \not{q}' (\not{x} \not{k} - \not{y} \not{p}) \not{\epsilon} (\not{x} \not{k} + (1-x) \not{p})] \right. \\ & \left. + \text{tr} [Y_5 (- (1-x) \not{x} - \not{y} \not{p}) \not{\epsilon}' \not{\epsilon} \not{\epsilon} (\not{x} \not{k} + (1-x) \not{p})] \right] \end{aligned}$$

what remains is to calculate traces, need
extract \not{q}_5 pole from 5th integral,
and do Feynman parameter integral.

~~$\text{Tr} [Y_5 \not{q}' \not{\epsilon}] = \text{tr} [Y_5 \not{k} \not{\epsilon}' \not{\epsilon}] = -4i E^{\mu\nu\rho\sigma} k_\mu \epsilon'_\nu P_\rho \epsilon_\sigma$~~

These traces can be done in 4 dimensions because

$(4-D) \int \frac{\not{e}_L}{(2\pi)^D} \frac{\not{e}^2}{(\not{e}' + D)^3}$ is ~~finite~~ finite in $D=4$ only

order $(D-4)$ terms in $\text{Tr} [\dots]$ vanish when we take
 $D \rightarrow 4$.

Also K_L anticommutes w/ $\not{q}, \not{p}, \not{k}, \not{\epsilon}'$

$$\{K_L, \not{q}\} \text{ e.g. } \{K_L, \not{p}\} = 2 \not{e}_L \cdot \not{p}_\parallel = 0.$$

(8)

If there is a finite mass term

$$\int d^4x \bar{\psi} i\cancel{D}\psi - m\bar{\psi}\psi$$

$$S \int d^4x \bar{\psi} i\cancel{D}\psi - m\bar{\psi}\psi = \int d^4x \alpha(x) \left(2\bar{j}_5^m - 2m\bar{\psi}\gamma_5\psi \right) \quad (1)$$

The anomaly can be viewed as a correction to one-loop effective action from loop diagrams involving the second term in (1):

$$= - \frac{(2\pi i)^2}{(4\pi)^2} (2im\gamma_5) \epsilon^{\mu\nu\alpha\beta} k^\mu \epsilon^\nu p^\alpha \epsilon^\beta$$

\Rightarrow at one loop

$$S \left(\int d^4x \bar{\psi} (i\cancel{D} - m)\psi \right) = \int d^4x \alpha \left[2\bar{j}_5^m + \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} - 2m\bar{\psi}\gamma_5\psi \right]$$

(7)

skip

further details + state answer

$$\int d^4x e^{-im\cdot x} \langle p, k | 2_m j_5^\mu | 0 \rangle$$

$$= i j_m \left(\text{curly loop} + \text{square loop} \right)$$

$$= \frac{e^2}{2\pi^2} \epsilon^{\mu\nu\rho\sigma} k_\mu \epsilon'_\nu p_\rho \epsilon_B$$

which F.T. of $\langle p, q | -\frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_\mu F_B | 0 \rangle$

$2_m j_5^\mu = -\frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_\mu F_B$	Adler-Bell-Jackiw anomaly
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chiral anomaly as failure of measure

to be invariant under chiral transformation

The path integral for chiral invariant theory of fermions is

$$Z = \int D\psi D\bar{\psi} e^{i \int d^4x \bar{\psi} i\cancel{D}\psi}$$

under $\psi \rightarrow e^{i\alpha \gamma_5} \psi$ $\bar{\psi} \rightarrow \bar{\psi} e^{i2\alpha \gamma_5}$ $\alpha \text{ const}$

$$\int d^4x \bar{\psi} i\cancel{D}\psi \rightarrow \int d^4x (\bar{\psi} i\cancel{D}\psi - 2m\alpha(\gamma) \bar{\psi} \gamma^5 \gamma_5 \psi)$$

$$= \int d^4x \bar{\psi} i\cancel{D}\psi - \alpha(x) \partial_\mu (\bar{\psi} \gamma^5 \gamma_5 \psi)$$

$$= \int d^4x \bar{\psi} i\cancel{D}\psi + \alpha(x) \partial_\mu \int_5^m$$

(9)

From 2 in the $m \rightarrow 0$ limit this anomaly term must be associated with

$$\text{Det}' D\bar{\psi}' = D\bar{\psi} \partial \bar{\psi} \cdot \text{Det}(e^{iB\bar{\psi}})$$

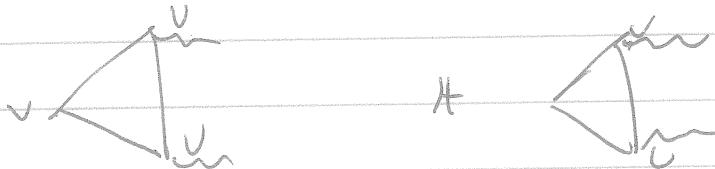
For functional evaluation of determinants
(see Peskin)

Another way to see the anomaly in 4-D is to go to Euclidean space and use.

$$\int d^4 p_E \left[f\left(\frac{p^m + q}{E}\right) - f\left(\frac{p^m}{E}\right) \right] = \int d^3 p \lim_{L \rightarrow \infty} \frac{q^m \delta m}{E} f\left(\frac{p}{E}\right)$$

anomaly results from surface term depends on linear shift. If we shift

$$p^m \rightarrow p^m + \alpha k^m + \beta p^m \quad \text{and shift}$$



$$\partial_n j_V^m = (1 - b_1 - b_2) - \frac{e^2}{8\pi^2} \tilde{F}^{mn} F_{nv}$$

$$\partial_n j_A^m = (1 - b_1 + b_2) - \frac{e^2}{8\pi^2} \tilde{F}^{mn} F_{nv}$$

we can choose b_1, b_2 to get rid of anomaly in j_V^m but not j_A^m

10

Noth renormalization - one-loop exact

$$N_R - N_L = \int \frac{d^3x}{8\pi^2} \frac{e^2}{2} F^{uv} F_{uv}$$

This must be integer if there were other corrections from h.o. diagrams this would no longer be true the case

Adler-Bardeen - all orders proof.



→ integral over hemispherical momentum only can be rendered sufficiently convergent so surface terms vanish.

Next applications

- $\pi^0 \rightarrow \gamma\gamma$

- anomaly cancellation in gauge theory

- B, L nonconservation in SM

- $U(1)$ anomaly in $\phi\psi\phi + \eta'$ meson

instantons, monopoles, topology ...