Final Exam
Phy 341 - Spring 2004
Assigned: Wed. April 21, 2004. **Due: Noon, Thursday April 29, 2004.**
All work is to be done independently. This is an open book exam. The problems are worth 20 points each. Please feel free to stop by my office or email me if you have questions.

Problem 1

Consider QED with two charged spin-$\frac{1}{2}$ fields, one of which is massive, the other massless. The Lagrangian is

$$\mathcal{L} = \overline{\psi}_1 i\not{D}\,\psi_1 + \overline{\psi}_2(i\not{D} - m)\psi_2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,.$$

where $D_\mu = \partial_\mu + ieA_\mu$.

a) Write down the equations of motion for each of the fields.

b) What are the symmetries of this theory? Specify whether they are global or gauge symmetries. Write down the conserved Noether currents corresponding to each symmetry. Verify using the equations of motion that these currents are conserved.

c) Suppose I take the limit $m \to 0$. Are there additional symmetries? If so, write down the corresponding Noether currents.

d) Suppose I add a term to the Lagrangian of the form

$$\mathcal{L}' = -e\,\overline{\psi}_2\not{A}\,\psi_1 \; - e\,\overline{\psi}_1\not{A}\,\psi_2\,.$$

Is this interaction gauge invariant? Verify your answer by checking whether or not the matrix element for the decay $\psi_2 \to \psi_1\gamma$ mediated by this interaction satisfies the Ward identity.

Problem 2

In class we developed the trace technology for computing unpolarized cross sections and decay rates making extensive use of the formulae

$$\sum_s u^s(p)\,\overline{u}^s(p) = \not{p} + m\,, \qquad \sum_s v^s(p)\,\overline{v}^s(p) = \not{p} - m\,.$$

Using trace technology to compute cross sections/decay rates for polarized particles requires the following generalization:

$$u^s(p)\,\overline{u}^s(p) = (\not{p} + m)\frac{1+\gamma_5\not{s}}{2}\,, \tag{1}$$

where the polarization four-vector s^μ is defined by

$$2m\,s^\mu = \overline{u}^s(p)\gamma^\mu\gamma^5 u^s(p)\,. \tag{2}$$

In this Problem we will verify Eq.(1) and apply it to heavy fermion decay. (The analogous formula for $v^s(p)\overline{v}^s(p)$ is not needed for this problem.)

a) First verify that the four-vector s^μ defined in Eq. (2) is the polarization vector of a spin-$\frac{1}{2}$ particle by showing that in the rest frame of the fermion, $p^\mu = (E, \vec{p}) = (m, \vec{0})$, the time-component of s^μ vanishes and space components are a unit vector pointing in the direction of the fermion spin. For example, if in the rest frame the fermion has $s_z = \pm 1/2$ then the four vector $s^\mu = (s^t, s^x, s^y, s^z) = (0, 0, 0, \pm 1)$.

b) Bilinears that can be constructed from $\overline{u}^s(p)$ and $u^s(p)$ are

$$
\begin{aligned}
\overline{u}^s(p) u^s(p) &= 2m \\
\overline{u}^s(p) \gamma^\mu \gamma_5 u^s(p) &= 2m\, s^\mu \\
\overline{u}^s(p) \gamma^\mu u^s(p) &= 2p^\mu \\
\overline{u}^s(p) \gamma_5 u^s(p) &= 0 \\
\overline{u}^s(p) \sigma^{\mu\nu} u^s(p) &= 2\epsilon^{\mu\nu\alpha\beta} s_\alpha p_\beta \,.
\end{aligned}
\tag{3}
$$

The first equation establishes the normalization of $u^s(p)$ and the second defines s^μ. Use the Dirac equation to verify the last three identities in Eq. (3).

c) Argue using the completeness of the gamma matrices that $u^s(p)\overline{u}^s(p)$ can be determined in terms of the bilinears in Eq. (3). Verify Eq. (1).

d) Consider a theory with a heavy fermion, ψ_2, of mass m, and massless fermion, ψ_1, and a scalar, ϕ, with mass $m_\phi (< m)$. These particles are coupled by the interaction Lagrangian

$$
\mathcal{L} = g\overline{\psi}_2 \left(\frac{1 - \gamma_5}{2} \right) \psi_1 \phi + \text{h.c.}\,.
$$

Calculate partial width for the decay $\psi_2 \to \psi_1 \phi$ as well as the angular distribution

$$
\frac{1}{\Gamma} \frac{d\Gamma}{d\cos\theta} \,,
$$

where $\cos\theta$ is the angle between the polarization vector of ψ_2 and the direction of the ψ_1 three-momentum. Can you explain your result for the angular distribution by considering the angular momentum of the inital and final states and applying elementary quantum mechanics?

Problem 3

Assuming the T-ordered product of two free scalar fields is given by $\langle 0|T\phi(x)\phi(0)|0\rangle = D(x)$, show that

$$
\langle 0|T e^{i\phi(x)} e^{-i\phi(0)}|0\rangle = e^{[D(x) - D(0)]} \,.
$$

(In general, $D(0)$ is divergent and requires regularization, but you need not worry about this.)

Problem 4

Consider the following theory of two scalar fields

$$
\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 + \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2) - \frac{\lambda}{4!}(\phi_1^2 + \phi_2^2)^2 \,.
$$

Notice the mass term has the wrong sign. This means that the classical vaccuum is not $\phi_1 = \phi_2 = 0$ but $\phi_1^2 + \phi_2^2 = v^2$, $v \neq 0$.

a) Sketch the vaccuum energy as a function of v and find the value of v that minimizes the energy.

In cases like this we must expand the fields around the true ground state. Define

$$\phi_1 = v + \rho, \phi_2 = \pi$$

and rewrite the Lagrangian in terms of the fields ρ and π. What are the masses of the fields ρ and π? One should be massless.

b) Write down all the Feynman rules for this theory (in terms of the fields ρ and π).

c) Calculate the scattering amplitude for $\pi\pi$ scattering to lowest order in perturbation theory. You should find that the amplitude vanishes as the external momenta of the π goes to zero.

Problem 5

In this problem you will use field theory methods to solve a nonrelativistic quantum mechanics problem. Consider a particle of unit mass moving in one-dimension in a potential that is the harmonic oscillator potential plus small anharmonic corrections. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - g x^3 - h x^4 \,.$$

Treating the anharmonic terms as a perturbation, the shift in the ground state energy to lowest order in g^2 and h is given by

$$\Delta E = \frac{3h}{4\omega^2} - \frac{11 g^2}{8\omega^4} \,.$$

You will reproduce this result by treating this system as a field theory in zero space and one time dimensions. The propagator for the x field is $i/(k^2 - \omega^2 + i\epsilon)$, where k is now a "one-momentum" (energy) rather than a four-momentum. For each loop you must integrate over $dk/2\pi$ rather than $d^4k/(2\pi)^4$. Feynman rules for the vertices are the same as 3+1 dimensional field theory and there are no UV divergences so you do not have to worry about renormalization. The shift in the ground state energy is given by a formula analogous to Eq. (4.56) of Peskin. There are three vacuum diagrams that contribute to the $O(h, g^2)$ correction in Eq. (4). One requires a evaluating a genuine two-loop integral which I will do for you:

$$\begin{aligned} I_2 &= \int \frac{dp}{2\pi}\frac{dk}{2\pi}\frac{i}{p^2 - \omega^2 + i\epsilon}\frac{i}{k^2 - \omega^2 + i\epsilon}\frac{i}{(p+k)^2 - \omega^2 + i\epsilon} \\ &= -i\int \frac{dp_E}{2\pi}\frac{dk_E}{2\pi}\frac{1}{p_E^2 + \omega^2}\frac{1}{k_E^2 + \omega^2}\frac{1}{(p_E + k_E)^2 + \omega^2} \\ &= \frac{-i}{12\,\omega^4} \,. \end{aligned}$$

The first step is the Wick rotation and the second integral can be done by repeatedly using the residue theorem.