Suggested Solutions – Problem Set 4, Phy 315

Problem 1. 10 points

a) $I_t = Uk$ in old basis $\{|n\rangle\}$

$$\langle n'|\, I_t\, |n\rangle$$
$$= \langle \nu'|\, W\, I_t\, W^+\, |\nu\rangle$$
$$= \langle \nu'|\, W\, U k\, W^+\, |\nu\rangle \qquad \searrow\ kW^+ = (W^+)^* k = W^T k$$
$$= \langle \nu'|\, W\, U\, W^T\, k\, |\nu\rangle$$
$$= \langle \nu'|\, \tilde{U} k\, |\nu\rangle$$

where $\tilde{U} = W U W^T$

b) $\tilde{I}_t^2 = \tilde{U} k \tilde{U} k$

$$= W U W^T k\, W U W^T k$$
$$= W U k W^+ W U k W^+$$
$$= W (Uk)(W^+W)(Uk) W^+ \qquad \searrow\ W^+W = 1$$
$$= W (Uk)^2 W^+$$
$$= W\, I_t^2\, W^+$$

Since $I_t^2 = \pm 1$,

$$\tilde{I}_t = W(\pm 1) W^+ = (\pm 1) W W^+ = \pm 1$$

Therefore $I_t^2 = \tilde{I}_t^2 = \pm 1$, not depend on the representation.

Problem 2 — 10 Point

a) $M(a_m \to b+c) = C(\lambda_b\ \lambda_c)\ D^J_{m\lambda}(\hat{n}) \qquad \lambda = \lambda_b - \lambda_c$

The initial state is a random mixture of states with all m, so all m's equally probable with $\frac{1}{2J+1}$ and we need sum over m.

$$\frac{d\Gamma}{d\Omega} \propto \frac{1}{2J+1} \sum_m |M(a_m \to b+c)|^2$$

$$\propto |C(\lambda_b, \lambda_c)|^2 \sum_m D^J_{m\lambda}(\hat{n})\ D^J_{m\lambda}(\hat{n})^*$$

$$D^J_{m\lambda}(\hat{n})^* = D^{J\,+}_{\lambda m}(\hat{n}) = D^J_{\lambda m}(-\hat{n})$$

$$\sum_m D^J_{\lambda m}(-\hat{n})\ D^J_{m\lambda}(\hat{n})$$

$$= \sum_m \langle \lambda |\ e^{i\vec{J}\cdot\vec{\theta}(\hat{n})}\ | m \rangle \langle m |\ e^{-i\vec{J}\cdot\vec{\theta}(\hat{n})}\ | \lambda \rangle$$

$$= \langle \lambda |\ e^{i\vec{J}\cdot\vec{\theta}(\hat{n})}\ e^{-i\vec{J}\cdot\vec{\theta}(\hat{n})}\ | \lambda \rangle$$

$$= \langle \lambda | \lambda \rangle$$

$$= 1$$

Therefore $\frac{d\Gamma}{d\Omega} \propto |C(\lambda_b, \lambda_c)|^2$ is independent of $\hat{n}$ and the angular distribution is isotropic.

b) If we specify m and sum over λ_b, λ_c,

$$\frac{d\Gamma}{d\Omega} \propto \sum_{\lambda_b \lambda_c} |C(\lambda_b, \lambda_c)|^2\ D^J_{m\lambda}(\hat{n})\ D^J_{m\lambda}(\hat{n})^*$$

$$= \sum_{\lambda_b \lambda_c} |C(\lambda_b, \lambda_c)|^2\ D^J_{m\lambda}(\hat{\lambda})\ D^J_{\lambda m}(-\hat{n})$$

$$= \langle m |\ e^{-i\vec{J}\cdot\vec{\theta}(\hat{n})}\ \underbrace{\sum_{\lambda_b, \lambda_c} |C(\lambda_b, \lambda_c)|^2\ |\lambda\rangle\langle\lambda|}_{\neq 1}\ e^{i\vec{J}\cdot\vec{\theta}(\hat{n})}\ | m \rangle$$

So in general, we can't cancel $e^{\pm i\vec{J}\cdot\vec{\theta}(\hat{n})}$ and the distribution depends on $\hat{n}$. then not isotropic.

Problem 3 — 10 Points

Angular distribution given by

$$\frac{d\Gamma}{d\Omega(\hat{n})} \propto \sum_{\lambda_b,\lambda_c} |c(\lambda_b,\lambda_c)|^2 \; D^J_{M\Lambda}(\hat{n})\, D^J_{\Lambda M'}(-\hat{n})\, \rho_{M'M}$$

$\rho_{M'M} = \langle M'|\rho|M\rangle$ where ρ is density operator.

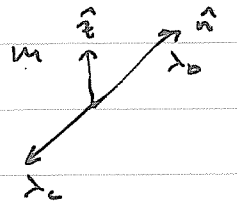
$$\rho = \sum_i P_i |i\rangle\langle i|$$

ρ is Hermitian and can always be diagonalized, it need not be diagonal in $\hat{z}$-basis, however.

Let us use freedom to choose axis of quantization so that $\rho_{M'M}$ is diagonal.

$$\frac{d\Gamma}{d\Omega(\hat{n})} \propto \sum_{\lambda_b,\lambda_c} |c(\lambda_b,\lambda_c)|^2 \; D^J_{M\Lambda}(\hat{n})\, D^J_{\Lambda M}(-\hat{n})\, \rho_{MM}$$

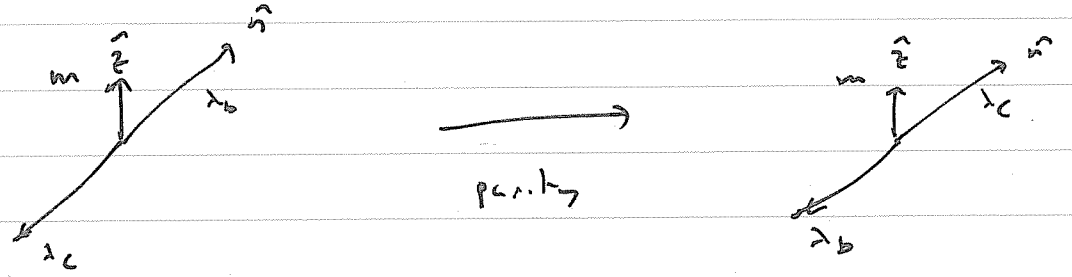
Then $D^J_{M\Lambda}(\hat{n})\, D^J_{\Lambda m}(-\hat{n})$ describes



initial state : $J_z = m$

final state : $J_{\hat{n}} = \lambda_b - \lambda_c = \Lambda$

the parity reflection reverses helicity, not spin



the final state now has helicity $-\Lambda$ along $\hat{n}$ axis

or " Λ along $-\vec{n}$ axis

$$\frac{d\Gamma}{d\Omega(-\hat{n})} = \sum_{\lambda_b,\lambda_c} |C(\lambda_b,\lambda_c)|^2 \; D^J_{M\Lambda}(-\hat{n})\, D^J_{\Lambda m}(\hat{n})\, \rho_{mm}$$

$$= \sum_{\lambda_b,\lambda_c} |C(\lambda_b,\lambda_c)|^2 \; D^J_{M-\Lambda}(\hat{n})\, D_{-\Lambda m}(-\hat{n})\, \rho_{mm}$$

but if interaction is parity invariant $C(\lambda_b,\lambda_c) = C(-\lambda_b,-\lambda_c)$

$$= \sum_{\lambda_b\lambda_c} |C(-\lambda_b,\lambda_c)|^2 \; D^J_{M-\Lambda}(\hat{n})\, D_{-\Lambda m}(-\hat{n})\, \rho_{mm}$$

$$= \sum_{\lambda_b,\lambda_c} |C(\lambda_b,\lambda_c)|^2 \; D^J_{M\Lambda}(\hat{n})\, D_{\Lambda m}(-\hat{n})\, \rho_{mm}$$

$$= \frac{d\Gamma}{d\Omega(\hat{n})} \quad \checkmark$$

Problem 4. 10 Points

a) First, verify $T_\kappa^{(K)} = \sum_{\kappa_1 \kappa_2} \langle K\kappa \mid k_1 \kappa_1 k_2 \kappa_2 \rangle U_{\kappa_1}^{(k_1)} U_{\kappa_2}^{(k_2)}$ is a tensor operator of rank K.

Plug this into eq. (237)

$D^+(R)\, T_\kappa^{(k)}\, D(R)$

$= D^+(R) \sum_{\kappa_1 \kappa_2} \langle K\kappa \mid k_1 \kappa_1 k_2 \kappa_2 \rangle U_{\kappa_1}^{(k_1)} U_{\kappa_2}^{(k_2)} D(R)\rangle$

$= \sum_{\kappa_1 \kappa_2} \langle K\kappa \mid k_1 \kappa_1 k_2 \kappa_2 \rangle \Big(D^+(R) U_{\kappa_1}^{(k_1)} \underbrace{D(R)\big)\big(D^+(R)}_{=1} U_{\kappa_2}^{(k_2)} D(R) \Big)$ ↓ (237)

$= \sum_{\kappa_1 \kappa_2} \langle K\kappa \mid k_1 \kappa_1 k_2 \kappa_2 \rangle \Big(\sum_{\kappa_1'} D_{\kappa_1 \kappa_1'}^{k_1} U_{\kappa_1'}^{(k_1)} \Big)\Big(\sum_{\kappa_2'} D_{\kappa_2 \kappa_2'}^{k_2} U_{\kappa_2'}^{(k_2)} \Big)$

$= \sum_{\kappa_1 \kappa_2} \sum_{\kappa_1' \kappa_2'} \langle K\kappa \mid k_1 \kappa_1 k_2 \kappa_2 \rangle D_{\kappa_1 \kappa_1'}^{k_1} D_{\kappa_2 \kappa_2'}^{k_2} U_{\kappa_1'}^{(k_1)} U_{\kappa_2'}^{(k_2)}$ ↓ (7.160)

$= \sum_{\kappa_1 \kappa_2} \sum_{\kappa_1' \kappa_2'} \langle K\kappa \mid k_1 \kappa_1 k_2 \kappa_2 \rangle \sum_{\kappa' \kappa'' \kappa'} \langle k_1 \kappa_1 k_2 \kappa_2 \mid K'\kappa' \rangle D_{\kappa' \kappa''}^{K'} \langle K'\kappa'' \mid k_1 \kappa_1' k_2 \kappa_2' \rangle U_{\kappa_1'}^{(k_1)} U_{\kappa_2'}^{(k_2)}$

Since $\sum_{\kappa_1 \kappa_2} \langle K\kappa \mid k_1 \kappa_1 k_2 \kappa_2 \rangle \langle k_1 \kappa_1 k_2 \kappa_2 \mid K'\kappa' \rangle = \delta_{KK'} \delta_{\kappa\kappa'}$ (3.177)

$D^+(R)\, T_\kappa^{(k)}\, D(R)$ becomes

$= \sum_{\kappa_1' \kappa_2' \kappa''} D_{\kappa\kappa''}^{K} \underbrace{\langle K\kappa'' \mid k_1 \kappa_1' k_2 \kappa_2' \rangle U_{\kappa_1'}^{(k_1)} U_{\kappa_2'}^{(k_2)}}$

$= \sum_{\kappa''} D_{\kappa\kappa''}^{K} T_{\kappa''}^{(k)}$

$= \sum_{\kappa'} D_{\kappa\kappa'}^{K} T_{\kappa'}^{(k)}$

Therefore $T_{\kappa'}^{(k)} = \sum_{\kappa_1 \kappa_2} \langle K\kappa \mid k_1 \kappa_1 k_2 \kappa_2 \rangle U_{\kappa_1}^{(k_1)} U_{\kappa_2}^{(k_2)}$ is a tensor operat

b) Verify (242)

r.h.s. $= \sum_{k\kappa} \langle k_1 \kappa_1 k_2 \kappa_2 | k \kappa \rangle T_\kappa^{(k)}$ ↓ (243)

$= \sum_{k\kappa} \langle k_1 \kappa_1 k_2 \kappa_2 | k \kappa \rangle \sum_{\kappa_1' \kappa_2'} \langle k \kappa | k_1 \kappa_1' k_2 \kappa_2' \rangle U_{\kappa_1'}^{(k_1)} U_{\kappa_2'}^{(k_2)}$

$= \sum_{\kappa_1' \kappa_2'} \langle k_1 \kappa_1 k_2 \kappa_2 \left(\sum_{k\kappa} |k\kappa\rangle\langle k\kappa| \right) k_1 \kappa_1' k_2 \kappa_2' \rangle U_{\kappa_1'}^{(k_1)} U_{\kappa_2'}^{(k_2)}$

$= \sum_{\kappa_1' \kappa_2'} \langle k_1 \kappa_1 k_2 \kappa_2 | k_1 \kappa_1' k_2 \kappa_2' \rangle U_{\kappa_1'}^{(k_1)} U_{\kappa_2'}^{(k_2)}$

$= \delta_{\kappa_1 \kappa_1'} \delta_{\kappa_2 \kappa_2'} U_{\kappa_1'}^{(k_1)} U_{\kappa_2'}^{(k_2)}$

$= U_{\kappa_1}^{(k_1)} U_{\kappa_2}^{(k_2)}$

Problem 5 — 10 Points

a) $$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}}\,\frac{(x \pm iy)^2}{r^2}$$

$$Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\,\frac{(x \pm iy)z}{r^2}$$

$$Y_2^{0} = \sqrt{\frac{5}{16\pi}}\,\frac{(3z^2 - r^2)}{r^2}$$

Therefore we can get:

$$xy = -i\sqrt{\frac{2\pi}{15}}\left(Y_2^{2} - Y_2^{-2}\right) r^2$$

$$xz = -\sqrt{\frac{2\pi}{15}}\left(Y_2^{1} - Y_2^{-1}\right) r^2$$

$$x^2 - y^2 = \sqrt{\frac{8\pi}{15}}\left(Y_2^{2} + Y_2^{-2}\right) r^2$$

b) $$Q \equiv e\langle \alpha, j, m | (3z^2 - r^2) | \alpha, j, m \rangle$$

$$= e\langle \alpha, j, m | \sqrt{\frac{16\pi}{5}}\, r^2 Y_2^0 | \alpha, j, m \rangle$$

$$= e\sqrt{\frac{16\pi}{5}}\,\langle \alpha, j, m | r^2 Y_2^0 | \alpha, j, m \rangle$$

$$= e\sqrt{\frac{16\pi}{5}}\,\frac{\langle \alpha, j \| r^2 Y_2^{(m)} | \alpha, j \rangle}{\sqrt{2j+1}}\,\langle j, m | 2, 0, j, m \rangle$$

Then, $$e\langle \alpha, j, m' | (x^2 - y^2) | \alpha, j, m \rangle$$

$$= e\langle \alpha, j, m' | \sqrt{\frac{8\pi}{15}}\, r^2 \left(Y_2^{2} + Y_2^{-2}\right) | \alpha, j, m \rangle$$

$$= e\sqrt{\frac{8\pi}{15}}\,\frac{\langle \alpha, j \| r^2 Y_2^{(m)} | \alpha, j \rangle}{\sqrt{2j+1}}\left[\langle j, m' | 2, 2, j, m \rangle + \langle j, m' | 2, -2, j, m \rangle\right]$$

$$= \frac{Q}{\sqrt{6}}\left\{\frac{\langle j, m' | 2, 2, j, m \rangle + \langle j, m' | 2, -2, j, m \rangle}{\langle j, m | 2, 0, j, m \rangle}\right\}$$

For $m = j$, $\langle j m' | 2, 2, j, j \rangle$ vanish

$$e\langle \alpha, j, m' | (x^2 - y^2) | \alpha, j, j \rangle = \frac{Q}{\sqrt{6}}\,\frac{\langle j, m' | 2, -2, j, m \rangle}{\langle j, m | 2, 0, j, m \rangle}$$