Problem Set 1 · solutions

#1

$$L = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} - \sum_i V(\vec{r}_i - \vec{r}_j)$$

$$\delta \vec{r}_i = -\vec{v}\, t \qquad \delta \dot{\vec{r}}_i = -\vec{v}$$

a) $$\delta L = -\sum_i m_i \dot{\vec{r}}_i \cdot \vec{v}$$

$$= \frac{d}{dt}\left(-\vec{v} \sum_i m_i \vec{r}_i\right) \qquad \boxed{\Omega = -\sum_i m_i \vec{r}_i \cdot \vec{v}}$$

b) $$G = \sum_i \vec{P}_i \cdot \delta \vec{r}_i - \Omega$$

$$= -\vec{v} \cdot \sum_i m_i \dot{\vec{r}}_i\, t + \vec{v} \cdot \sum_i m_i \vec{r}_i$$

$$= \vec{v} \cdot \left(\sum_i m_i \vec{r}_i - \sum_i m_i \dot{\vec{r}}_i\, t \right)$$

$\vec{P} = \sum_i m_i \dot{\vec{r}}_i$ - total momentum

$\vec{R}_{cm} = \dfrac{\sum_i m_i \vec{r}_i}{\sum_i M_i} = \dfrac{\sum_i m_i \vec{r}_i}{M}$ - center-of-mass

$M = \sum_i m_i$ - total mass

$$G = \vec{v} \cdot \left(M \vec{R}_{cm} - \vec{P} t \right) \qquad \boxed{M \vec{R}_{cm} - \vec{P} t \text{ is conserved}}$$

Since $\dot{\vec{P}} = 0$ by translation invariance we also have

$$M \dot{\vec{R}}_{cm} = \vec{P}$$

c) $$\{\vec{r}_i, G\} = \sum_j \left(\frac{\partial \vec{r}_i}{\partial \vec{r}_j} \cdot \frac{\partial G}{\partial \vec{p}_j} - \cancel{\frac{\partial \vec{r}_i}{\partial \vec{p}_j}}^{0} \cdot \frac{\partial G}{\partial \vec{r}_j} \right)$$

$$= \sum_j \delta_{ij} \, \delta \vec{r}_j$$

$$= - \vec{v}_i t \quad \checkmark$$

$$\{\vec{p}_i, G\} = \sum_j \cancel{\frac{\partial \vec{p}_i}{\partial \vec{r}_j}}^{0} \cdot \frac{\partial G}{\partial \vec{p}_j} - \frac{\partial \vec{p}_i}{\partial \vec{p}_j} \cdot \frac{\partial G}{\partial \vec{r}_j}$$

$$= - \sum_j \delta_{ij} \, \vec{v} m_j$$

$$= - m_i \vec{v} \quad \checkmark$$

$\boxed{\#2}$ (use $\hbar = 1$ units)

a) $e^{i\vec{L}\cdot\vec{\theta}}\, r_i\, e^{-i\vec{L}\cdot\vec{\theta}}$

$= r_i + i\left[\vec{L}\cdot\vec{\theta}, r_i\right]$

$= r_i + i\theta_j\left[L_j, r_i\right]$

$= r_i - \theta_j \epsilon_{jik} r_k \qquad \epsilon_{jik} = -\epsilon_{ijk}$

$= r_i + (\vec{\theta}\times\vec{r})_i$

or $\vec{r}' = \vec{r} + \vec{\theta}\times\vec{r}$

b) key formula is

$$e^{A} B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{3!}[A[A,[A,B]]] + \cdots$$

$[iL_z, x] = -y \qquad [iL_z, y] = x \qquad [iL_z, z] = 0$

$\Rightarrow \quad e^{iL_z\theta}\, z\, e^{-iL_z\theta} = z$

$e^{iL_z\theta}\, x\, e^{-iL_z\theta} = x - \theta y - \frac{1}{2}\theta^2 x + \frac{1}{3!}\theta^3 y + \cdots$

$= x\left(1 - \frac{\theta^2}{2} + \cdots\right) - y\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$

$= x\cos\theta - y\sin\theta$

similarly $e^{iL_z\theta}\, y\, e^{-iL_z\theta} = x\sin\theta + y\cos\theta$

#3 $[L_i, A_j] = i\hbar \epsilon_{ijk} A_k$

immediately follows from the fact that $\vec{A}$ is a vector. Equivalently, it can be proven from repeated application of

$$[L_i, r_j] = i\hbar \epsilon_{ijk} r_k$$
$$[L_i, p_j] = i\hbar \epsilon_{ijk} p_k$$
$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$$

The next 5 pages contain my notes on evaluating

$$[A_i, A_j] \quad \text{and} \quad \vec{A}\cdot\vec{A}$$

In these notes, I have reversed the sign of $\vec{A}$ and rescaled by a factor of m:

$$\vec{A} \to -\vec{A}/m$$

relative to what was given in class.

$$\vec{A} = \frac{(\vec{p}\times\vec{L}) - (\vec{L}\times\vec{p})}{2m} - e^2\frac{\vec{r}}{r}$$

$$(\vec{L}\times\vec{p})_i = \varepsilon_{ijk} L_j p_k = \varepsilon_{ijk}\left([L_j, p_k] + p_k L_j\right)$$
$$= \varepsilon_{ijk}\, i\hbar\, \varepsilon_{jkb} p_b - (\vec{p}\times L)_i$$
$$= 2i\hbar\, p_i - (\vec{p}\times\vec{L})_i$$

$$\boxed{\vec{A}_i = \frac{(\vec{p}\times\vec{L})_i}{m} - i\hbar\frac{p_i}{m} - e^2\frac{r_i}{r}}$$

$$[A_i, A_j] = \left[\frac{(p\times L)_i}{m} - \frac{i\hbar p_i}{m} - \frac{e^2 r_i}{r},\ \frac{(p\times L)_j}{m} - i\hbar\frac{p_j}{m} - \frac{e^2 r_j}{r}\right]$$

$$= \frac{1}{m^2}[(p\times L)_i, (p\times L)_j] - \frac{i\hbar}{m^2}\left([(p\times L)_i, p_j] + [p_i, (p\times L)_j]\right)$$
$$- \frac{e^2}{m}\left(\left[(p\times L)_i, \frac{r_j}{r}\right] + \left[\frac{r_i}{r}, (p\times L)_j\right]\right) + \frac{i\hbar e^2}{m}\left(\left[p_i, \frac{r_j}{r}\right] + \left[\frac{r_i}{r}, p_j\right]\right)$$

$$[(p\times L)_i, (p\times L)_j] = \varepsilon_{iab}\varepsilon_{jcd}[p_a L_b, p_c L_d]$$

$$= \varepsilon_{iab}\varepsilon_{jcd}\left(p_a [L_b, p_c L_d] + [p_a, p_c L_d] L_b\right)$$
$$= \varepsilon_{iab}\varepsilon_{jcd}\left(p_a [L_b, p_c] L_d + p_a p_c [L_b, L_d] + p_c [p_a, L_d] L_b\right)$$

$$= i\hbar\,\varepsilon_{iab}\varepsilon_{jcd}\left(\varepsilon_{bcf}\, p_a p_f L_d + p_a p_c\, \varepsilon_{bdf} L_f + \varepsilon_{adf}\, p_c p_f L_b\right)$$

$$= i\hbar\,\varepsilon_{iab}\Big[(\delta_{bj}\delta_{df} - \delta_{jf}\delta_{bd}) p_a p_f L_d + (\delta_{jf}\delta_{bc} - \delta_{fc}\delta_{jb}) p_a p_c L_f$$
$$+ (\delta_{jf}\delta_{ac} - \delta_{aj}\delta_{cf}) p_c p_f L_b\Big]$$

$$= i\hbar \Big[\cancel{\varepsilon_{iaj}\, p_a\, \vec{p}\cdot\vec{L}} - \cancel{\varepsilon_{iab}\, p_a\, p_j\, L_b} + \cancel{\varepsilon_{iab}\, p_a\, p_b}\, L_j - \cancel{\varepsilon_{iaj}\, p_a\, \vec{p}\cdot\vec{L}}$$

$$+ \cancel{\varepsilon_{iab}\, p_a\, p_j\, L_b} - \varepsilon_{ijb}\, p^2 L_b \Big]$$

$$= \boxed{-i\hbar\, \varepsilon_{ijb}\, p^2 L_b = \left[(p\times L)_i, (p\times L)_j \right]}$$

$$\left[(p\times L)_i, p_j \right] + \left[p_i, (p\times L)_j \right]$$

$$\left[(p\times L)_i, p_j \right] - (i \leftrightarrow j)$$

$$= \varepsilon_{iab} \left[p_a L_b, p_j \right] - (i \leftrightarrow j)$$

$$= \varepsilon_{iab}\, p_a\, i\hbar\, \varepsilon_{bjk}\, p_k - (i \leftrightarrow j)$$

$$= i\hbar \left(\delta_{ij}\, p^2 - p_i p_j \right) - (i \leftrightarrow j)$$

$$\underline{= 0}$$

$$\left[(p\times L)_i, \frac{r_j}{r} \right] + \left[\frac{r_i}{r}, (p\times L)_j \right]$$

$$\varepsilon_{iab} \left[p_a L_b, \frac{r_j}{r} \right] - (i \leftrightarrow j)$$

$$= \varepsilon_{iab} \left[\left[p_a, \frac{r_j}{r} \right] L_b + p_a \left[L_b, r_j \right] \frac{1}{r} - (i \leftrightarrow j) \right]$$

$$= \varepsilon_{iab} \left[-i\hbar\, p_a \left(\frac{r_j}{r} \right) L_b + i\hbar\, p_a\, \varepsilon_{bjk}\, r_k \frac{1}{r} - (i \leftrightarrow j) \right]$$

$$= \varepsilon_{iab} \left[-i\hbar \left(\frac{\delta_{aj}}{r} - \frac{r_a r_j}{r^3} \right) L_b + i\hbar\, p_a\, \varepsilon_{bjk}\, r_k \frac{1}{r} - (i \leftrightarrow j) \right]$$

$$= -i\hbar \frac{\varepsilon_{ijk}L_k}{r} + i\hbar \frac{r_j (r\times L)_i}{r^3} + i\hbar\left(\cancel{\delta_{ij}\vec{p}\cdot\vec{r}} - p_j r_i\right)\frac{1}{r} - (i\leftrightarrow j)$$

$$= -2i\hbar\, \varepsilon_{ijk}L_k \frac{1}{r} + i\hbar\,(p_i r_j - p_j r_i)\frac{1}{r} + i\hbar \frac{1}{r^3}\left[r_j (r\times L)_i - r_i (r\times L)_j\right]$$

$$r_j (r\times L_i) - (i\leftrightarrow j) = \varepsilon_{iab}\, r_j r_a L_b - (i\leftrightarrow j)$$
$$= \varepsilon_{iab}\varepsilon_{bcd}\, r_j r_a r_c p_d - (i\leftrightarrow j)$$
$$= (\delta_{ic}\delta_{ad} - \delta_{id}\delta_{ca})\, r_j r_a r_c p_d - (i\leftrightarrow j)$$
$$= r_i r_j\, \vec{r}\cdot\vec{p} - r^2 r_j p_i - (i\leftrightarrow j)$$
$$= r^2 (r_i p_j - r_j p_i)$$

$$= -2i\hbar\, \varepsilon_{ijk}L_k \frac{1}{r} + \cancel{i\hbar\,(p_i r_j - p_j r_i)\frac{1}{r} + i\hbar\frac{1}{r}(r_i p_j - r_j p_i)}$$

$$\boxed{= -2i\hbar\, \varepsilon_{ijk}L_k \frac{1}{r} = \left[(p\times L)_i, \frac{r_j}{r}\right] + \left[\frac{r_i}{r}, (p\times L)_j\right]}$$

$$\left[p_i, \frac{r_j}{r}\right] + \left[\frac{r_i}{r}, p_j\right] = -i\hbar\, \nabla_i \frac{r_j}{r} - (i\leftrightarrow j) = -i\hbar\left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3}\right) - (i\leftrightarrow j) = \underline{0}$$

$$[A_i, A_j] = -\frac{i\hbar}{m^2}\varepsilon_{ijk}\, p^2 L_k + 2\frac{e^2}{m} i\hbar\, \varepsilon_{ijk}L_k \frac{1}{r}$$

$$[A_i, A_j] = -\frac{2i\hbar}{m}\left(\frac{p^2}{2m} - \frac{e^2}{r}\right)\varepsilon_{ijk}L_k$$

$$\boxed{[A_i, A_j] = -2\frac{\hat{H}}{m}\, i\hbar\, \varepsilon_{ijk}L_k}$$

Computing $\vec{A}\cdot\vec{A}$

$$\vec{A}\cdot\vec{A} = \left[\frac{(\vec{p}\times\vec{L})}{m} - i\hbar\frac{\vec{p}}{m} - \frac{e^2}{r}\vec{r}\right]\cdot\left[\frac{(\vec{p}\times\vec{L})}{m} - i\hbar\frac{\vec{p}}{m} - \frac{e^2\vec{r}}{r}\right]$$

$$= \frac{(\vec{p}\times\vec{L})\cdot(\vec{p}\times\vec{L})}{m^2} - i\hbar\,\frac{\vec{p}\cdot(\vec{p}\times\vec{L}) + (\vec{p}\times\vec{L})\cdot\vec{p}}{m^2} - \frac{\hbar^2\vec{p}^2}{m^2}$$

$$-\frac{e^2}{m}\left(\frac{1}{r}\,\vec{r}\cdot(\vec{p}\times\vec{L}) + (\vec{p}\times\vec{L})\cdot\vec{r}\,\frac{1}{r}\right) + \frac{i\hbar e^2}{m}\left(\vec{p}\cdot\vec{r}\,\frac{1}{r} + \frac{1}{r}\,\vec{r}\cdot\vec{p}\right) + e^4$$

$$(\vec{p}\times\vec{L})\cdot(\vec{p}\times\vec{L}) = \varepsilon_{iab}\varepsilon_{icd}\,P_a L_b P_c L_d$$
$$= (\delta_{ac}\delta_{bd} - \delta_{ad}\cancel{\delta_{bc}})\,P_a L_b P_c L_d$$
$$= P_a L_b P_a L_b$$
$$= P^2L^2 + P_a[L_b, P_a]L_b$$
$$= \underline{P^2L^2} + i\hbar\,\varepsilon_{bac}\,\cancel{P_a}^{\,0}P_c L_b$$

$$\vec{p}\cdot(\vec{p}\times\vec{L}) = 0 \qquad (\vec{p}\times\vec{L})\cdot\vec{p} = \varepsilon_{iab}P_a L_b P_i$$
$$= \varepsilon_{iab}P_a[L_b, P_i] + \varepsilon_{iab}\cancel{P_a P_i}^{\,0} L_b$$
$$= i\hbar\,\varepsilon_{iab}\varepsilon_{bij}\,P_a P_j$$
$$= \underline{2i\hbar\,P^2}$$

<u>1st 3 terms are:</u> $$= \frac{P^2L^2}{m^2} + \frac{2\hbar^2}{m^2}P^2 - \frac{\hbar^2P^2}{m^2} = (L^2+\hbar^2)\frac{P^2}{m^2}$$

$$\vec{r}\cdot(\vec{p}\times\vec{L}) = \varepsilon_{ijk}r_iP_jL_k = L^2$$

~~$(\vec{p}\times\vec{L})\cdot\vec{r} = \varepsilon_{ijk}P_iL_jr_k = i\hbar\,\varepsilon_{ijk}\,\varepsilon_{jkl}$~~

$$(\vec{p}\times\vec{L})\cdot\vec{r} = \varepsilon_{iab}\, p_a L_b r_i = \varepsilon_{iab}\, p_a\left([L_b, r_i] + r_i L_b\right)$$

$$= \varepsilon_{iab}\, p_a\left(i\hbar\, \varepsilon_{bid} r_d + r_i L_b\right)$$

$$= 2i\hbar\, \vec{p}\cdot\vec{r} + L^2$$

4th + 5th terms are:

$$-\frac{e^2}{m}\left(\frac{2}{r}L^2\right) - \frac{2i\hbar e^2}{m}\,\vec{p}\cdot\vec{r}\,\frac{1}{r} + \frac{i\hbar e^2}{m}\left(\vec{p}\cdot\vec{r}\,\frac{1}{r} + \frac{1}{r}\,\vec{r}\cdot\vec{p}\right)$$

$$= -\frac{2e^2}{mr}L^2 + \frac{i\hbar e^2}{m}\,\delta^{ij}\left[\frac{r_i}{r}, p_j\right]$$

$$= -\frac{2e^2}{mr}L^2 - \frac{\hbar^2 e^2}{m}\,\delta^{ij}\,\nabla_j \frac{r_i}{r}$$

$$= -\frac{2e^2}{mr}L^2 - \frac{\hbar^2 e^2}{m}\,\delta^{ij}\left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3}\right)$$

$$= -\frac{2e^2}{mr}\left(L^2 + \hbar^2\right)$$

$$\vec{A}\cdot\vec{A} = \frac{2}{m}\left(\frac{p^2}{2m} - \frac{e^2}{r}\right)\left(L^2 + \hbar^2\right) + e^4$$

$$\boxed{\vec{A}\cdot\vec{A} = \frac{2\hat{H}}{m}\left(L^2 + \hbar^2\right) + e^4}$$

On subspace of states w/ fixed E.

$$\vec{A}\cdot\vec{A} = \frac{2E}{m}\left(L^2 + \hbar^2\right) + e^4$$

#4

a) $e^{i(\vec{L}\cdot\vec{\theta}+\vec{K}\cdot\vec{\theta}')} R^2 e^{-i(\vec{L}\cdot\vec{\theta}+\vec{K}\cdot\vec{\theta}')} = R^2$

$\Rightarrow \quad [L_i, R^2] = [K_i, R^2] = 0$

let $\quad R^2 = x_j x_j + w^2$

$[L_i, R^2] = [L_i, x_j x_j] = 2x_j [L_i, x_j]$

$= 2i\, x_j\, \epsilon_{ijk} x_k = 0$

$[K_i, R^2] = [w p_i - x_i p_w, x_j x_j + w^2]$

$= w[p_i, x_j x_j] - x_i [p_w, w^2]$

$= 2x_j w [p_i, x_j] - 2x_i w [p_w, w]$

$= -i2wx_i + i2x_i w$

$= 0$

b) $[L_i, K_j] = [L_i, w p_j - x_j p_w]$

$= w[L_i, p_j] - [L_i, x_j] p_w$

$= i\epsilon_{ijk} w p_k - i\epsilon_{ijk} x_k p_w$

$= i\epsilon_{ijk} K_k$

$$[K_i, K_j] = [w p_i - x_i p_w, \; w p_j - x_j p_w]$$

$$= - [w p_i, x_j p_w] - [x_i p_w, w p_j]$$

$$= - w [p_i, x_j] p_w - x_j p_i [w, p_w] - x_i p_j [p_w, w]$$

$$- w p_w [x_i, p_j]$$

$$= i \delta_{ij} \cancel{w p_w} - i x_j p_i + i x_i p_j - i \cancel{w p_w} \delta_{ij}$$

$$= i (x_i p_j - x_j p_i)$$

$$\left(\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right)$$

$$= i \epsilon_{ijk} \epsilon_{klm} x_l p_m$$

$$\boxed{[K_i, K_j] = i \epsilon_{ijk} L_k}$$