

Problem Set 6

Phy 315 - Fall 2006

Assigned: Thursday, November 9 Due: Tuesday, November 21, 5 pm

Problem 1: Green's Function for Partial Waves

a) Evaluate the free particle Green's function in the angular momentum basis:

$$\begin{aligned}\langle \vec{x} | \frac{1}{E - H_0 + i\epsilon} | \vec{x}' \rangle &= 2m \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{q}\cdot\vec{x} - i\vec{q}\cdot\vec{x}'}}{k^2 - q^2 + i\epsilon} \\ &= -i 2m k \sum_{lm} j_l(kr_<) h_l(kr_>) Y_l^{m*}(\hat{x}) Y_l^m(\hat{x}').\end{aligned}$$

Use the plane wave decomposition (Abers, Eq. 8.82) to rewrite the exponentials in terms of spherical harmonics. The angular part of the d^3q integral can be evaluated using the orthogonality of spherical harmonics. The final integral over the magnitude of \vec{q} can be done via contour integration. Useful identities:

$$j_l(x) = \frac{h_l(x) + h_l^*(x)}{2} \quad h_l(-x) = (-)^l h_l^*(x) \quad h_l(\rho) \xrightarrow{\rho \rightarrow \infty} \frac{e^{i\rho}}{i^{l+1}\rho}.$$

b) Using your result from part a), the Lippmann-Schwinger equation, and the expansions in Eqs. 8.97 and 8.98 of Abers, derive the radial integral equation (Eq. 8.103 of Abers).

Problem 2: Inelastic Scattering and Optical Theorem

a) Show that if a potential absorbs particles so the scattering is not elastic, the asymptotic form of the radial wavefunction for the l th partial wave is

$$R_l(r) \xrightarrow{r \rightarrow \infty} (-)^{l+1} \frac{e^{-ikr}}{r} + S_l(k) \frac{e^{ikr}}{r}$$

where $S_l(k) = \eta_l(k) e^{2i\delta_l}$. The inelasticity factor, $\eta_l(k)$, is real and $\eta_l(k) < 1$.

By considering probability currents, show that

$$\begin{aligned}\sigma_{inel} &= \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)(1 - \eta_l^2) \\ \sigma_{el} &= \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)(1 + \eta_l^2 - 2\eta_l \cos 2\delta_l),\end{aligned}$$

and verify the optical theorem

$$\sigma_{tot} = \frac{4\pi}{k} \text{Im} f(0),$$

where $f(0)$ is the forward scattering amplitude.

b) Is it possible to have inelastic scattering without elastic scattering? Why or why not?

Problem 3: Resonances and Square Well Potential

a) In the vicinity of a resonance, the phase shift varies rapidly from 0 to π . This can be modelled by the following ansatz for the phase shift:

$$\delta_l = \delta_b + \tan^{-1} \left(\frac{E - E_0}{\Gamma/2} \right) + \frac{\pi}{2}, \quad (1)$$

where δ_b is a slowly varying “background” phase shift. The last two terms are ≈ 0 for $|E - E_0| \gg \Gamma/2, E < E_0$ and $\approx \pi$ for $E - E_0 \gg \Gamma/2, E > E_0$. Show that if $\delta_b \approx 0$, the cross section in the vicinity of the resonance is well approximated by the Breit-Wigner form

$$\sigma_l \approx \frac{4\pi}{k^2} (2l + 1) \frac{(\Gamma/2)^2}{(E - E_0)^2 + (\Gamma/2)^2}.$$

b) Show that in the vicinity of the resonance $S_l(k)$ can be written as

$$S_l(k) = e^{2i\delta_l} \approx \frac{E - E_0 - i\Gamma/2}{E - E_0 + i\Gamma/2}.$$

In class, we showed that bound states correspond to poles in the scattering amplitude at imaginary values of momentum. Where in the complex momentum plane are the poles of the scattering amplitude associated with resonances? Give a heuristic argument as to why $1/\Gamma$ can be interpreted as the lifetime of the resonant state.

c) The S-wave phase shift for an attractive spherical square well of depth V_0 and range r_0 is

$$\delta_0 = -kr_0 + \tan^{-1} \left(\frac{k}{k'} \tan k'r_0 \right),$$

where k' is the wavenumber inside the potential, i.e. $k' = \sqrt{2m(E + V_0)}$. In this problem we consider low energy scattering, $kr_0 \ll 1$, and study how δ_0 varies as the depth of the potential well, or equivalently k' , is varied. Show that whenever $k' \approx k'_n = (2n + 1)\pi/(2r_0)$, δ_0 takes on the resonant form in Eq. (1). Show that k'_0 corresponds to the well developing its first bound state at zero energy. As the well is made deeper, this level moves down and at k'_1 it develops another zero-energy bound state, etc.

Problem 4: Resonances and δ -function potential

Consider scattering off a **repulsive** δ -function potential

$$V(r) = \frac{\hbar^2 \gamma}{2m} \delta(r - R) \quad \gamma > 0.$$

a) Write down the transcendental equation that determines the S-wave phase shift.

b) Assume γ is large, i.e.

$$\gamma \gg \frac{1}{R}, k.$$

Show that if $\tan kR$ is **not** close to zero the S-wave phase shift resembles that of the hard sphere result $\delta_0 = -kR$. (The hard sphere result is derived in 8.4.3 of Abers or 7.6 of

Sakurai.) Show that when the $\tan kR$ is close to zero resonance behavior is possible, i.e. $\cot \delta_0$ goes through zero as k increases. Determine approximately the positions of the resonances keeping terms of order $1/\gamma$; compare them with bound state energies for a particle confined inside a spherical wall of the same radius: $V(r) = 0, r < R, V(r) = \infty, r > R$. Obtain an approximate expression for the resonance width Γ , defined by

$$\Gamma = \frac{-2}{[d(\cot \delta_0)/dE]|_{E=E_r}}.$$

What happens to the resonance width as γ becomes large?

Problem 5: Discrete Symmetries and Scattering

a) Show that the scattering amplitude can be written as

$$f(\vec{k}, \vec{k}') = -4\pi^2 m \langle \vec{k}' | T | \vec{k} \rangle,$$

where T is the transition operator

$$\begin{aligned} T &= V + V \frac{1}{E - H_0 + i\epsilon} T \\ &= V + V \frac{1}{E - H_0 + i\epsilon} V + V \frac{1}{E - H_0 + i\epsilon} V \frac{1}{E - H_0 + i\epsilon} V + \dots, \end{aligned}$$

and $|\vec{k}\rangle$ and $|\vec{k}'\rangle$ are momentum eigenkets.

b) Show that if the time reversal and parity operator commute with $H = H_0 + V$, then the parity operator commutes with T , while for time reversal we have $\Theta^{-1}T\Theta = T^\dagger$, where Θ is the time-reversal operator. Use this result to derive the principle of **detailed balance**:

$$\frac{d\sigma}{d\Omega}(\vec{k} \rightarrow \vec{k}') = \frac{d\sigma}{d\Omega}(\vec{k}' \rightarrow \vec{k}).$$

Problem 6: Indistinguishable Particles and Scattering

a) Consider the elastic scattering of identical bosons. Argue that the scattering wavefunction must take on the form

$$e^{i\vec{k}\cdot\vec{x}} + e^{-i\vec{k}\cdot\vec{x}} + [f(\theta) + f(\pi - \theta)] \frac{e^{ikr}}{r},$$

where $\vec{x} = \vec{x}_1 - \vec{x}_2$ is the relative position of the two particles. Show that the differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta) + f(\pi - \theta)|^2.$$

How much larger is the differential cross section at $\theta = \pi/2$ relative to the case of distinguishable particles, assuming the same $f(\theta)$?

a) Now consider the elastic scattering of identical spin-1/2 fermions from two **unpolarized** beams. If the beams are unpolarized beams then the initial states are a statistical

mixture of 1/4 spin-0 states and 3/4 spin-1 states. Show that the differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |f(\pi - \theta)|^2 - \text{Re}|f(\theta)f^*(\pi - \theta)|. \quad (2)$$

How much smaller is differential cross section for fermions at $\theta = \pi/2$ relative to that of bosons, given the same $f(\theta)$? Use Eq. (2) to write down the differential cross section for the Coulomb scattering of identical electrons.