Problem Set 2
Phy 315 - Fall 2006
Assigned: Thursday, September 14 **Due:** Tuesday, September 26

Tinkham, Chap. 2: Abstract Group Theory

Problems 2.1, 2.2, 2.4

Tinkham, Chap. 3: Group Representations

Problems 3.1, 3.2, 3.3, 3.4

Problem 8: Cubic Symmetry Group

The finite subgroup of the rotation group which leaves a cube invariant is S_4, which contains 24 elements. Let C_m denote a rotation by $2\pi/m$, and let NC_m label a class consisting of N C_m rotations. The classes are:

i) E - the identity,
ii) $8C_3$ - 8 rotations by $2\pi/3$ about axes along the diagonals of the cube,
iii) $3C_2$ - 3 rotations by π about axes perpendicular to face centers of the cube,
iv) $6C_2$ - 6 rotations by π about axes that pass through midpoints of opposite edges of the cube,
v) $6C_4$ - 6 rotations by $\pi/4$ about axes perpendicular to face centers of the cube.

The character table of S_4 is

	E	$8C_3$	$3C_2$	$6C_2$	$6C_4$
Γ_1	1	1	1	1	1
Γ_2	1	1	1	-1	-1
Γ_3	2	-1	2	0	0
Γ_4	3	0	-1	-1	1
Γ_5	3	0	-1	1	-1

We have labelled the representations Γ_i, $i = 1, ..., 5$.

a) Explain why the angle of the rotation must be the same for rotations in the same class.
b) Why are the C_2 rotations described in iii) and iv) above in different classes?

Problem 9: Splitting of Degenerate States of an Atom in Cubic Crystal

Suppose an atom has a single outer electron in an angular momentum eigenstate with total angular momentum $\hbar^2 l(l+1)$. This atom has $2l+1$ degeneracy. Now place the atom in an external field with cubic symmetry. For example, the atom could be in a solid with a cubic crystal lattice, and the field is due to neigboring atoms in the crystal. The $2l+1$ states that were degenerate in free space will form a (possibly reducible) representation of the cubic symmetry group.

a) Write down the general expression for the character for the $2l+1$ dimensional representation of S_4.

b) Explicitly evaluate the character for $l = 0, 1, 2, 3$. Determine which irreps of the cubic symmetry group appear in each of the representations. In each case, explain how the energy levels split, i.e., how many distinct energy levels will the formerly degenerate $2l+1$ states split into and with what degeneracy.

Problem 10: Selection Rules

In this problem, you will find selection rules for the electric dipole operator between states that transform as irreducible representations of S_4. Consider the matrix elements

$$\langle \Gamma_i | \, \vec{r} \, | \Gamma_j \rangle \, ,$$

where $|\Gamma_i\rangle$ represents a state that transforms in the Γ_i irreducible representation.

a) Explain why the operators $\vec{r}$ transform in the irreducible representation Γ_4.

b) Use the character table to determine the tensor decomposition $\Gamma_4 \times \Gamma_i$, $i = 1, 2, 3, 4, 5$. Determine selection rules for the matrix element given above.