Problem 1 (15 pts.)

Consider a particle moving in one dimension whose Lagrangian is

$$L = \frac{1}{2} \dot{x}^2 - \frac{g}{x^2}.$$ 

a) Show that the following infinitesimal coordinate transformations are symmetries:

<table>
<thead>
<tr>
<th>Transformation</th>
<th>$\delta x$</th>
<th>$\delta \dot{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>time translation</td>
<td>$\epsilon \dot{x}$</td>
<td>$\epsilon \ddot{x}$</td>
</tr>
<tr>
<td>dilatation</td>
<td>$\epsilon \left(-\frac{1}{2} x + t \dot{x}\right)$</td>
<td>$\epsilon \left(\frac{1}{2} \dot{x} + t \ddot{x}\right)$</td>
</tr>
<tr>
<td>conformal transformation</td>
<td>$\epsilon (-t x + t^2 \dot{x})$</td>
<td>$\epsilon (-x + t \dot{x} + t^2 \ddot{x})$</td>
</tr>
</tbody>
</table>

b) Use Noether’s procedure to find the conserved quantity associated with each of these symmetries. Denote the conserved quantities associated with dilatations and conformal transformations by $D$ and $K$, respectively. Use the classical equations of motion to confirm that $H$ (the Hamiltonian), $D$, and $K$ are constants of the motion.

c) Write down the quantum mechanical operators corresponding to $H$, $D$, and $K$ in terms of the quantum mechanical operators $\hat{x}$ and $\hat{p}$. Compute the commutators

$$[H, D] = ??, \quad [H, K] = ??, \quad [D, K] = ??.$$ 

(Hint: this calculation can be considerably simplified by exploiting the fact that $H$, $D$, and $K$ are constants of the motion.) Are $H$, $D$, and $K$ generators of a Lie group?

d) Using the commutators computed in part c), show that expectation values of $H$, $D$ and $K$ are time independent in the quantum theory.
Problem 2 (15 pts.)

Consider a scattering experiment in which an electron with helicity $+\frac{1}{2}$ moving in the $+\hat{z}$ direction annihilates with a positron with helicity $-\frac{1}{2}$ moving along the $-\hat{z}$ direction. The final state consists of a muon moving in the $\hat{n}$ direction and an antimuon moving in the $-\hat{n}$ direction. The scattering angle, $\theta$, is defined by $\cos \theta = \hat{n} \cdot \hat{z}$. Assuming the interactions are rotationally invariant, calculate the angular distribution when

a) the final state consists of a muon with helicity $+\frac{1}{2}$ and an antimuon with helicity $-\frac{1}{2}$,
b) the final state consists of a muon with helicity $-\frac{1}{2}$ and an antimuon with helicity $+\frac{1}{2}$,
c) the helicities of the final state particles are unobserved, and the interaction allows the final state in part a) with probability amplitude $\propto \sqrt{p}$ and the final state in part b) with probability amplitude $\propto \sqrt{1-p}$. For what value of $p$ is the angular distribution forward-backward symmetric (i.e. invariant under $\theta \rightarrow \pi - \theta$). Explain why.

Problem 3 (10 pts.)

Consider electrons confined to a 2-dimensional surface, with coordinates $x$ and $y$, in the presence of a uniform magnetic field in the $z$-direction with strength $B$.

a) Write down a Lagrangian for a single electron in this situation.

The energy gap between the lowest Landau level (LLL) and the first excited state is $\hbar \omega_c = \hbar eB/mc$, where $m$ is the mass of the electron. By taking the limit $\hbar \omega_c \rightarrow \infty$, we can force all electrons into the LLL. Such a limit might be used to describe a situation in which there is insufficient energy to excite any electron into the excited Landau levels. The limit $\hbar \omega_c \rightarrow \infty$ can be obtained by either taking $B \rightarrow \infty$ or $m \rightarrow 0$. Apply this limit to the Lagrangian derived in part a) and show that it becomes

$$L_{LLL} = \frac{e}{c} B \dot{x} \dot{y}.$$  

b) Using $L_{LLL}$, find the momentum canonical to $x$, and determine the commutation relation:

$$[x, y] = ???.$$

c) Derive an uncertainty principle for $\Delta x \Delta y$ using the commutator derived in part b). Interpret this uncertainty relation by applying semiclassical quantization to the classical orbits of a particle in a uniform $B$ field.
Problem 4 (10 pts.)

Below is the character table of a discrete group called $A_4$. The symmetry group has 4 classes. The identity is its own class, denoted by $E$. Other classes consist of rotations and are denoted $NC_i$, where each rotation in the class is a rotation about $2\pi/i$ and $N$ is the number of rotations in the class. In the table below, $\omega = e^{2\pi i/3}$.

$$
\begin{array}{c|ccc}
\Gamma & E & 3C_2 & 4C_3 & 4C'_3 \\
\hline
\Gamma_1 & 1 & 1 & 1 & 1 \\
\Gamma_2 & 1 & 1 & \omega & \omega^2 \\
\Gamma_3 & 1 & 1 & \omega^2 & \omega \\
\Gamma_4 & 1 & 1 & 1 & 1 \\
\end{array}
$$

a) Fill in the entries in the character table which have been left blank.

b) Suppose an atom with a single electron is placed in a crystal with $A_4$ symmetry. Let the electron have orbital angular momentum $l$, where $l = 0, 1, 2, \text{ or } 3$. For each case, explain how the $2l + 1$ states of split into the representations, $\Gamma_i$, and give the degeneracy of each representation.

c) Let $\vec{d} = \sum_i e \vec{x}_i$ be the electric dipole operator, determine the selection rules for the matrix element $\langle \Gamma_a | \vec{d} | \Gamma_b \rangle$. 