Midterm
Phy 315 - Fall 2006
Assigned: Thursday, November 2, **Due: Tuesday, November 7, 5 pm**

TAKE HOME MIDTERM RULES:

• The exam is **open book**. You may consult the class textbooks, Gottfried and Yan, and Tinkham, and the course webpage. **No other sources.**
• You may consult your own past homeworks and your own class notes. **You may not borrow homework or notes from your colleagues.**
• You may not discuss **any aspect** of the midterm with anyone.
• Submit a signed pledge that you have abided by these rules as well as the Duke Community Standard (http://www.integrity.duke.edu/graduate/commstd.html).

Problem 1 (15 pts.)

Consider a particle moving in one dimension whose Lagrangian is

$$L = \frac{1}{2}\dot{x}^2 - \frac{g}{x^2}\,.$$

a) Show that the following infinitesmal coordinate transformations are symmetries:

$$\begin{aligned}
\text{time translation}: &\quad \delta x = \epsilon\,\dot{x} &\quad \delta\dot{x} = \epsilon\,\ddot{x}\\
\text{dilatation}: &\quad \delta x = \epsilon\left(-\frac{1}{2}x + t\,\dot{x}\right) &\quad \delta\dot{x} = \epsilon\left(\frac{1}{2}\dot{x} + t\,\ddot{x}\right)\\
\text{conformal transformation}: &\quad \delta x = \epsilon\,(-t\,x + t^2\,\dot{x}) &\quad \delta\dot{x} = \epsilon\,(-x + t\,\dot{x} + t^2\ddot{x})\,.
\end{aligned}$$

b) Use Noether's procedure to find the conserved quantity associated with each of these symmetries. Denote the conserved quantities associated with dilatations and conformal transformations by D and K, respectively. Use the classical equations of motion to confirm that H (the Hamiltonian), D, and K are constants of the motion.

c) Write down the quantum mechanical operators corresponding to H, D, and K in terms of the quantum mechanical operators $\hat{x}$ and $\hat{p}$. Compute the commutators

$$[H, D] = ?? \qquad [H, K] = ?? \quad [D, K] = ??\,.$$

(Hint: this calculation can be considerably simplified by exploiting the fact that H, D, and K are constants of the motion.) Are H, D, and K generators of a Lie group?

d) Using the commutators computed in part c), show that expectation values of H, D and K are time independent in the quantum theory.

Problem 2 (15 pts.)

Consider a scattering experiment in which an electron with helicity $+\frac{1}{2}$ moving in the $+\hat{z}$ direction annihilates with a positron with helicity $-\frac{1}{2}$ moving along the $-\hat{z}$ direction. The final state consists of a muon moving in the $\hat{n}$ direction and an antimuon moving in the $-\hat{n}$ direction. The scattering angle, θ, is defined by $\cos\theta = \hat{n}\cdot\hat{z}$. Assuming the interactions are rotationally invariant, calculate the angular distribution when

a) the final state consists of a muon with helicity $+\frac{1}{2}$ and an antimuon with helicity $-\frac{1}{2}$,

b) the final state consists of a muon with helicity $-\frac{1}{2}$ and an antimuon with helicity $+\frac{1}{2}$,

c) the helicities of the final state particles are unobserved, and the interaction allows the final state in part a) with probability amplitude $\propto \sqrt{p}$ and the final state in part b) with probability amplitude $\propto \sqrt{1-p}$. For what value of p is the angular distribution forward-backward symmetric (i.e. invariant under $\theta \to \pi - \theta$). Explain why.

Problem 3 (10 pts.)

Consider electrons confined to a 2-dimensional surface, with coordinates x and y, in the presence of a uniform magnetic field in the z-direction with strength B.

a) Write down a Lagrangian for a single electron in this situation.

The energy gap between the lowest Landau level (LLL) and the first excited state is $\hbar\omega_c = \hbar eB/(mc)$, where m is the mass of the electron. By taking the limit $\hbar\omega_c \to \infty$, we can force all electrons into the LLL. Such a limit might be used to describe a situation in which there is insufficient energy to excite any electron into the excited Landau levels. The limit $\hbar\omega_c \to \infty$ can be obtained by either taking $B \to \infty$ or $m \to 0$. Apply this limit to the Lagrangian derived in part a) and show that it becomes

$$\mathcal{L}_{LLL} = \frac{e}{c} B\dot{x}y\,.$$

b) Using $\mathcal{L}_{LLL}$, find the momentum canonical to x, and determine the commutation relation:

$$[x, y] = ???\,.$$

c) Derive an uncertainty principle for $\Delta x\,\Delta y$ using the commutator derived in part b). Interpret this uncertainty relation by applying semiclassical quantization to the classical orbits of a particle in a uniform B field.

Problem 4 (10 pts.)

Below is the character table of a discrete group called A_4. The symmetry group has 4 classes. The identity is its own class, denoted by E. Other classes consist of rotations and are denoted NC_i, where each rotation in the class is a rotation about $2\pi/i$ and N is the number of rotations in the class. In the table below, $\omega = e^{2\pi i/3}$.

	E	$3C_2$	$4C_3$	$4C_3'$
Γ_1	1	1	1	1
Γ_2	1	1	ω	ω^2
Γ_3	1	1	ω^2	ω
Γ_4				

a) Fill in the entries in the character table which have been left blank.

b) Suppose an atom with a single electron is placed in a crystal with A_4 symmetry. Let the electron have orbital angular momentum l, where $l = 0, 1, 2$, or 3. For each case, explain how the $2l + 1$ states of split into the representations, Γ_i, and give the degeneracy of each representation.

c) Let $\vec{d} = \sum_i e\vec{x}_i$ be the electric dipole operator, determine the selection rules for the matrix element $\langle \Gamma_a | \vec{d} | \Gamma_b \rangle$.