Final Exam  
Phy 315 - Fall 2006  
Assigned: Thursday, Dec 14, Due: Sunday, December 17, 5 pm

TAKE HOME FINAL RULES:

- The exam is **open book**. You may consult the following sources  
  - class texts: Gottfried and Yan, Tinkham;  
  - the course webpage.  
  **You may consult no other sources.**

- You may consult your own past homeworks and your notes. **You may not borrow or exchange homework or notes with your colleagues after the exam period begins.**  
- You may not discuss any aspect of the midterm with anyone.  
- Submit a signed pledge that you have abided by these rules as well as the Duke Community Standard (http://www.integrity.duke.edu/graduate/commstd.html).

Problem 1 (15 points)

Consider a quantum mechanical system described by the following Lagrangian:

\[ L = \text{Tr}[\dot{U}^\dagger \dot{U}] = \sum_{a,b=1}^{2} \dot{U}_{ab}^\dagger \dot{U}_{ba}, \]

where \( U \) is a \( 2 \times 2 \) special unitary matrix.

a) Show that the momentum conjugate to the coordinate \( U_{ab} \) is \( P_{ab} = \dot{U}_{ab}^\dagger = \dot{U}_{ba} \) and that the Hamiltonian is

\[ H = \text{Tr}[P^\dagger P]. \]

b) Show the equations of motion are \( \ddot{U}_{ab} = \ddot{U}_{ab}^\dagger = 0. \)

c) Show that the Lagrangian is invariant under the following symmetry transformations:

\[ U \rightarrow LU \quad U \rightarrow UR \]

where \( L \) and \( R \) are time independent unitary \( 2 \times 2 \) matrices.

d) Use the Noether procedure to find the associated conserved quantities :

\[ (J_L)_{ab} = \frac{1}{i} (U \dot{U})_{ab} \quad (J_R)_{ab} = i(U^\dagger \dot{U})_{ab}. \]

(The factors of \( i \) ensure that the \( (J_L)_{ab} \) and \( (J_R)_{ab} \) are hermitian, and the normalization of \( (J_L)_{ab} \) and \( (J_R)_{ab} \) is arbitrary. I have chosen them for convenience in what follows.) Using the equations of motion verify that \( (J_L)_{ab} \) and \( (J_R)_{ab} \) are conserved. (You will need to use the unitarity constraint \( U^\dagger U = U U^\dagger = 1. \))

e) The canonical commutation relations are

\[ [U_{ab}, P_{cd}] = [U_{ab}, \dot{U}_{dc}^\dagger] = i\delta_{ac}\delta_{db}. \]
Define

\[ J_L^i = \text{Tr}[\frac{\sigma^i}{2} J_L] \quad J_R^i = \text{Tr}[\frac{\sigma^i}{2} J_R] \]

and use the canonical commutations relations to verify

\[
\begin{align*}
[J_L^i, J_L^j] &= i\epsilon^{ijk} J_k^L \\
[J_R^i, J_R^j] &= i\epsilon^{ijk} J_k^R \\
[J_L^i, J_R^j] &= 0.
\end{align*}
\]

f) Rewrite the Hamiltonian in terms of \( J_L \) and \( J_R \). Use the result of part e) to find the energy eigenvalues and their degeneracy.

**Problem 2 (15 points)**

Consider an electron interacting with a potential which admits a shallow bound state. Shallow means that if the range of the potential is \( R \) and the binding energy of the bound state is \( E = \hbar^2 \gamma^2/(2m_e) \), then \( \gamma \ll 1/R \). At low energies, the S-wave scattering phase shift, \( \delta_0 \), is well approximated by

\[ k \cot \delta_0 = -\frac{1}{a} + \frac{r_0}{2} k^2. \]

Here \( r_0 \) is of order \( R \) but \( a \gg r_0 \).

a) What is the energy of the bound state in terms of the parameters \( a \) and \( r_0 \)?

b) What is the asymptotic form of the bound state wavefunction for \( r \gg R \)? You should be able to determine the \( r \) dependence but not necessarily the overall normalization.

c) Use the asymptotic form of the wavefunction obtained in b) to calculate the photo-electric dissociation differential cross section for the shallow bound state. Let the photon frequency be \( \omega \), the photon polarization vector be \( \vec{e} \), and the electron’s final momentum be \( \vec{k}_f \).
Problem 3 (15 points)

Consider 5 identical ions placed in a plane at the corners of a regular pentagon. This configuration is invariant under
- five reflections about an axis bisecting the pentagon;
- four rotations by \( \pm 2\pi/5 \) and \( \pm 4\pi/5 \);
- the identity operation,
so the order of the group is ten.

a) Work out the character table of the symmetry group of the pentagon.

b) Now imagine an atom with a single electron outside a closed shell in a D-wave is placed at the center of the pentagon. Explain how the five-fold degeneracy is lifted. State which representations of the pentagon symmetry group the electron can be in, and what their degeneracies are.

c) Now suppose the electron is in the state \( |l, m = 0\rangle \), where \( l \) is the orbital angular momentum quantum number and \( m \) is the eigenvalue of \( L_z \), where \( L_z \) is the component of angular momentum perpendicular to the plane of the pentagon. Explain why this state must lie in a one-dimensional representation of the pentagon symmetry group, and determine which one-dimensional representation for arbitrary \( l \).

Problem 4 (15 points)

Consider the orbital and spin angular momentum operators for the electromagnetic field. See, for example, Eq. (10.53-10.54) of Gottfried and Yan. Naively, we would expect that

\[
\begin{align*}
[L_i, A_j(\vec{r})] &= i(\vec{r} \times \vec{\nabla})_i A_j(\vec{r}) \\
[S_i, A_j(\vec{r})] &= i\epsilon_{ijk} A_k(\vec{r})
\end{align*}
\]

Here I have set \( \hbar = 1 \), \( L_i \) is the \( i \)th component of orbital angular momentum (\( J_{\text{orb}} \) in Gottfried-Yan’s notation) and \( S_i \) is the \( i \)th component of spin angular momentum (\( J_{\text{sp}} \) in Gottfried-Yan’s notation). The first commutation relation states that \( L_i \) generates a rotation of the coordinate \( \vec{r} \), while \( S_i \) generates a rotation of the vector index of \( A_i \).

However, in fact Eq. (1) is wrong and the actual commutation relations are

\[
\begin{align*}
[L_i, A_j(\vec{r})] &= i(\vec{r} \times \vec{\nabla})_i A_j(\vec{r}) + i\nabla_j f(\vec{A}) \\
[S_i, A_j(\vec{r})] &= i\epsilon_{ijk} A_k(\vec{r}) - i\nabla_j f(\vec{A})
\end{align*}
\]

The interpretation of Eq. (2) is that \( L_i \) and \( S_i \) generate both rotations and gauge transformations. Note that the second term cancels in the sum so that the total angular momentum operator, \( J_i = L_i + S_i \), generates the expected transformation.

a) Find \( f(\vec{A}) \) by computing one of the two commutation relations in Eq. (2). (I think \( [S_i, A_j(\vec{r})] \) is easier but I will leave this up to you. You need not compute both.)

b) The gauge condition we used is \( \vec{\nabla} \cdot \vec{A} = 0 \). Show that the second term in the commutation relation is necessary so that

\[
[S_i, \vec{\nabla} \cdot \vec{A}] = [L_i, \vec{\nabla} \cdot \vec{A}] = 0.
\]