Problem Set 5

Phy 211 - Fall 2003

Assigned: Friday, Fri. Oct. 3 Due: Thursday, October 9

Problem 1: Measurements in a 3 Dimensional Hilbert Space

Shankar, 4.2.1

Problem 2: Expectation Values of the Momentum Operator

Shankar, 4.2.2, 4.2.3

Problem 3: Commutation Relations

In a) and b), A, B, C and D are all operators.

a) Show that

$$\begin{aligned}[AB, C] &= A[B, C] + [A, C]B\,, \\ [AB, CD] &= A[B, C]D + AC[B, D] + C[A, D]B + [A, C]DB\,.\end{aligned}$$

b) Show that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots\,.$$

(Hint: It is easy to verify the first few terms in the series by expanding the left hand side in A. To prove the relation, consider the operator $f(\lambda) = e^{\lambda A} B e^{-\lambda A}$. First show that $df(\lambda)/d\lambda = [A, f(\lambda)]$. Use this to derive a recursion relation for the coefficients in the Taylor Series of $f(\lambda)$. Then set $\lambda = 1$.)

c) Let $\hat{x}, \hat{p}$ denote the position and momentum operators. Show that

$$[\hat{x}^n, \hat{p}] = i\hbar\, n\, \hat{x}^{n-1} \qquad [\hat{x}, \hat{p}^n] = i\hbar\, n\, \hat{p}^{n-1}\,.$$

and

$$[f(\hat{x}), \hat{p}] = i\hbar f'(\hat{x}) \qquad [\hat{x}, f(\hat{p})] = i\hbar f'(\hat{p})$$

Problem 4: Operators that Commute with the Hamiltonian

Formally solve the time-dependent Schrödinger equation by writing

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle\,.$$

Now suppose an observable, $\hat{O}$, commutes with the Hamiltonian: $[\hat{O}, \hat{H}] = 0$. Show that the expectation values of this quantity are time-independent, i.e.

$$\langle\hat{O}(t)\rangle \equiv \langle\psi(t)|\hat{O}|\psi(t)\rangle = \langle\psi(0)|\hat{O}|\psi(0)\rangle = \langle\hat{O}(0)\rangle\,.$$

Problem 5: Campbell-Baker-Hausdorff Formula (Graduate Student Problem)

Let A and B be two operators which do not commute ($[A, B] \neq 0$), but which commute with their commutator, i.e. $[A, [A, B]] = [B, [A, B]] = 0$. Show that

$$e^{A+B} = e^A e^B e^{-[A,B]/2}\,.$$