Problem Set 4
Phy 211 - Fall 2003
Assigned: Friday, Sept. 26 Due: Thursday, October 2

Problem 1: Diagonalization of Hermitian Matrices

Shankar, 1.8.2, 1.8.3

Problem 2: Diagonalization of Degenerate Hermitian Matrices

Shankar, 1.8.10

Problem 3: Determinants, Traces in Terms of Eigenvalues

Shankar, 1.8.6, 1.8.7

Problem 4: More on Hermitian, Unitary Matrices

a) Show that the determinant of a unitary matrix is a complex number of modulus unity.
b) Let $U = \exp(i\,H)$. Show that U is unitary if H is Hermitian. Using the results of Problem 3, show that

$$\mathrm{Det}\, U = \exp(i\,\mathrm{Tr}\, H)\,.$$

Problem 5: Dirac Notation

Suppose a particle is constrained to move on a circle. Its wave function, $\psi(\theta)$, must be a periodic function: $\psi(\theta + 2\pi) = \psi(\theta)$. Define the operators

$$\hat{\theta}\,\psi(\theta) = \theta\,\psi(\theta) \qquad\qquad \hat{p}_\theta\,\psi(\theta) = -i\frac{d}{d\theta}\,\psi(\theta)\,.$$

Eigenkets of these operators are defined by:

$$\begin{aligned} \hat{\theta}|\theta\rangle &= \theta|\theta\rangle & \langle\theta|\theta'\rangle &= \delta(\theta-\theta') \\ \hat{p}_\theta|n\rangle &= n|n\rangle & \langle n|m\rangle &= \delta_{nm}\,, \end{aligned}$$

where n, m are integers.
a) Why are the eigenvalues of $\hat{p}_\theta$ integers? Find the normalized eigenfunctions $\langle\theta|n\rangle$.
b) Starting with the completeness relation ($\mathbb{1}$ is the identity operator)

$$\sum_{n=-\infty}^{\infty} |n\rangle\langle n| = \mathbb{1}\,,$$

derive

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\theta}e^{-in\theta'} = \delta(\theta-\theta')\,.$$

c) Write the following equations in Dirac notation:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}e^{-im\theta} = \delta_{nm} \qquad \psi(\theta) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\psi_n\, e^{in\theta}\,.$$

Problem 6: Projection Operators (Graduate Student Problem)

a) Show that the eigenvalues of a projection operator ($P^2 = P$) are either 1 or 0. What are the possible values of $\mathrm{Tr}\, P$ in an n-dimensional vector space?

b) Suppose $\Omega^{-1}\,\Omega = I$, where I is the identity. Show that $\Omega\,\Omega^{-1}$ is a projection operator.

c) By computing $\mathrm{Tr}\,(\Omega^{-1}\,\Omega)$ and $\mathrm{Tr}\,(\Omega\,\Omega^{-1})$, prove that $\Omega\,\Omega^{-1} = I$ in a **finite** dimensional vector space. Why doesn't this proof work for infinite dimensional vector spaces? Can you construct a counter example?