Short Answer Questions

Let $|i\rangle, |j\rangle$ be orthonormal basis vectors in a vector space of dimension $n$.

1. (4 pts.) What is $\langle i|j\rangle$? What is $\sum_{i=1}^{n} |i\rangle\langle i|$?

2. (4 pts.) Suppose $|v\rangle = \sum_{i=1}^{n} v_i|i\rangle$.

Express $v_i$ as an inner product. Write $\langle v|$ as an expansion in basis bras (dual basis vectors).

3. (6 pts.) Let $\hat{X} = |i\rangle\langle j|.$

a) If $i \neq j$, is $\hat{X}$ Hermitian? Unitary? A projection operator? Explain.

4. (3 pts.) Let $|\psi'\rangle = \hat{O}|\psi\rangle$. State the condition that $\hat{O}$ must satisfy in order for the $|\psi'\rangle$ to be an orthonormal basis.

5. (3 pts.) What constraint must operators that represent physical observables in quantum mechanics obey? Why?

For questions 6-7, consider a particle moving in one-dimension, $x$, which ranges from $-\infty < x < \infty$. Let $|x\rangle$ be the eigenstates of the position operator: $\hat{x}|x\rangle = x|x\rangle$. The momentum operator is $\hat{p}$, and its eigenvectors are defined by $\hat{p}|p\rangle = p|p\rangle$.

6. (4 pts.) What is $\langle x|y\rangle$? State the completeness relation for this basis.

7. (6 pts.) a) If $\langle x|\psi\rangle \equiv \psi(x)$ what is $\langle x|\hat{p}|\psi\rangle$ in terms of $\psi(x)$?

b) Use this result to determine $\langle x|\hat{p}\rangle$.
   (NOTE: You need only determine the dependence on $x$, not the normalization.)
Problem 1. Two-dimensional Quantum Mechanical System

Suppose we have a quantum mechanical system described by a two dimensional vector space spanned by the basis vectors \(|1\rangle, |2\rangle\). Let \(\hat{H}\) be the Hamiltonian for this system. The matrix elements of \(\hat{H}\) are

\[
\langle 1 | \hat{H} | 1 \rangle = \langle 2 | \hat{H} | 2 \rangle = a \\
\langle 1 | \hat{H} | 2 \rangle = \langle 2 | \hat{H} | 1 \rangle = b .
\]

There is another observable for this system called \(\hat{S}\) whose matrix elements are

\[
\langle 1 | \hat{S} | 1 \rangle = 1 \\
\langle 2 | \hat{S} | 2 \rangle = -1 \\
\langle 1 | \hat{S} | 2 \rangle = \langle 2 | \hat{S} | 1 \rangle = 0 .
\]

Represent the basis vectors \(|1\rangle, |2\rangle\) as

\[
|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .
\]

a) (8 pts.) Write 2 x 2 matrices representing the operators \(\hat{H}\) and \(\hat{S}\) in this basis. What are the eigenvalues and eigenvectors of \(\hat{S}\)? What are the eigenvalues and eigenvectors of \(\hat{H}\)?
b) (7 pts.) Suppose the state vector of the system is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle).$$

State the possible outcomes and probabilities of a measurement of $\hat{H}$.

State the possible outcomes and probabilities of a measurement of $\hat{S}$. 
Problem 2. Time Evolution

a) (5 pts.) What are the stationary states of the Hamiltonian in the previous problem?

b) (5 pts.) Suppose the system is not in a stationary state. Do you expect the expectation values $\langle \hat{S} \rangle$ to be time dependent or constant in time. What about $\langle \hat{H} \rangle$?

c) (Graduate Student problem) (10 pts.) Suppose at $t = 0$ the system is in the state $|\psi(0)\rangle = |I\rangle$. What is the state at $|\psi(t)\rangle$? Compute $\langle \psi(t)|\hat{S}|\psi(t)\rangle$ to verify your answer to the previous question.