**Midterm 2**
Phy 211 - Fall 2002
Thursday, November 14

NO GRADUATE PROBLEMS. Max. undergrad score: 30 pts., max. grad score: 40 pts.
THERE ARE 2 PAGES TO THIS EXAM.

## Problem 1

Consider a particle in a box of length $L$ with a $\delta$-function in the middle:

$$\begin{aligned} V(x) &= \infty & |x| \geq L/2 \\ &= V_0\,\delta(x) & |x| \leq L/2 \end{aligned}$$

a) (2 pts.) What is the general solution to the time-independent Schrödinger equation in the regions $x < 0$ and $x > 0$?

b) (3 pts.) What are the boundary conditions at $x = 0, \pm L/2$?

c) (2 pts.) Show that the **odd** parity eigenfunctions of this Hamiltonian are the same as the odd parity eigenfunctions of the particle in a box **without** a $\delta$-function.

d) (3 pts.) For the **even** parity eigenfunctions, obtain a transcendental equation for the allowed wavenumbers. Show that the equation you obtain reproduces the allowed wavenumbers for a particle in a box when $V_0 \to 0$.

## Problem 2

In this problem, assume the Hamiltonian is that of a **free** particle,

$$\hat{H} = \frac{\hat{p}^2}{2m}\,.$$

a) (3 pts.) Use Ehrenfest's Theorem to find $\langle \hat{x}(t) \rangle$ in terms of $\langle \hat{x}(0) \rangle$ and $\langle \hat{p}(0) \rangle$.

b) (4 pts.) Let $\hat{x}_H(t)$ be the position operator in the **Heisenberg** picture. Define the pictures to be equivalent at $t = 0$, i.e. $\hat{x}_H(0) = \hat{x}$, where $\hat{x}$ is the Schrödinger picture (time-independent) operator. Evaluate the commutator

$$[\hat{x}_H(t), \hat{x}_H(0)]\,.$$

c) (3 pts.) Use the result of b) to derive an uncertainty relation for measurements of position **at different times**:

$$\Delta x(t) \Delta x(0)\,.$$

## Problem 3

In this problem we will consider a simple harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2.$$

The energy eigenkets and eigenvalues are defined by $\hat{H}|n\rangle = E_n|n\rangle$.

a) (4 pts.) Show that

$$\sum_{n'} |\langle n \mid \hat{x} \mid n' \rangle|^2 \, (E_{n'} - E_n) = \frac{\hbar^2}{2m} \, .$$

Let $U(\eta)$ be the **unitary** operator

$$U(\eta) = \exp[-\eta \, \hat{a}^2 + \eta \, (\hat{a}^\dagger)^2] \, ,$$

which satisfies the following identities:

$$\begin{aligned} U^\dagger(\eta) \, (\hat{a} + \hat{a}^\dagger) \, U(\eta) &= e^{2\eta} (\hat{a} + \hat{a}^\dagger) \\ U^\dagger(\eta) \, (\hat{a} - \hat{a}^\dagger) \, U(\eta) &= e^{-2\eta} (\hat{a} - \hat{a}^\dagger) \end{aligned} \tag{1}$$

b) (3 pts.) Calculate $\Delta x$ and $\Delta p$ for the state $|\eta\rangle = U(\eta)|0\rangle$. Is this a minimum uncertainty state?

c) (3 pts.) Verify Eq. (1). (Hint: Differentiate both sides with respect to $\eta$.)

## Problem 4

Consider a dumbbell constrained to rotate in the $x - y$ plane about its center of mass.

The Hamiltonian is given by

$$\hat{H} = \frac{\hat{L}_z^2}{2I} \, ,$$

where $I$ is the moment of inertia of the dumbbell.

a) (3 pts.) What are the allowed energy eigenvalues and what is their degeneracy? Suppose the dumbbell's wavefunction at time $t = 0$ is given by

$$\psi(\theta, t = 0) = \frac{1}{\sqrt{2\pi}} \left( \sqrt{\frac{1}{3}} e^{i\theta} + i \sqrt{\frac{2}{3}} e^{i2\theta} \right) \, .$$

b) (2 pts.) What is $\langle E \rangle$?

c) (5 pts.) What is the probability density of finding the dumbbell at angle $\theta$ at time $t$, (i.e. what is $|\psi(\theta, t)|^2$)?