Midterm 2  
Phy 211 - Fall 2002  
Thursday, November 14

NO GRADUATE PROBLEMS. Max. undergrad score: 30 pts., max. grad score: 40 pts.  
THERE ARE 2 PAGES TO THIS EXAM.

Problem 1

Consider a particle in a box of length \( L \) with a \( \delta \)-function in the middle:

\[
V(x) = \begin{cases}  
\infty & |x| \geq L/2 \\
V_0 \delta(x) & |x| \leq L/2 
\end{cases}
\]

a) (2 pts.) What is the general solution to the time-independent Schrödinger equation in the regions \( x < 0 \) and \( x > 0 \)?

b) (3 pts.) What are the boundary conditions at \( x = 0, \pm L/2 \)?

c) (2 pts.) Show that the odd parity eigenfunctions of this Hamiltonian are the same as the odd parity eigenfunctions of the particle in a box without a \( \delta \)-function.

d) (3 pts.) For the even parity eigenfunctions, obtain a transcendental equation for the allowed wavenumbers. Show that the equation you obtain reproduces the allowed wavenumbers for a particle in a box when \( V_0 \to 0 \).

Problem 2

In this problem, assume the Hamiltonian is that of a free particle,

\[
\hat{H} = \frac{\hat{p}^2}{2m}.
\]

a) (3 pts.) Use Ehrenfest’s Theorem to find \( \langle \dot{x}(t) \rangle \) in terms of \( \langle \dot{x}(0) \rangle \) and \( \langle \dot{p}(0) \rangle \).

b) (4 pts.) Let \( \hat{x}_H(t) \) be the position operator in the Heisenberg picture. Define the pictures to be equivalent at \( t = 0 \), i.e. \( \hat{x}_H(0) = \hat{x} \), where \( \hat{x} \) is the Schrödinger picture (time-independent) operator. Evaluate the commutator

\[
[\hat{x}_H(t), \hat{x}_H(0)].
\]

c) (3 pts.) Use the result of b) to derive an uncertainty relation for measurements of position at different times:

\[
\Delta x(t) \Delta x(0).
\]

Problem 3

In this problem we will consider a simple harmonic oscillator with Hamiltonian

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2.
\]

The energy eigenkets and eigenvalues are defined by \( \hat{H}|n\rangle = E_n|n\rangle.\)
a) (4 pts.) Show that
\[ \sum_{n'} |\langle n | \hat{x} | n'\rangle|^2 (E_{n'} - E_n) = \frac{\hbar^2}{2m}. \]

Let \( U(\eta) \) be the \textbf{unitary} operator
\[ U(\eta) = \exp[-\eta \hat{a}^2 + \eta (\hat{a}^\dagger)^2], \]
which satisfies the following identities:
\[ U^\dagger(\eta) (\hat{a} + \hat{a}^\dagger) U(\eta) = e^{2\eta}(\hat{a} + \hat{a}^\dagger) \]
\[ U^\dagger(\eta) (\hat{a} - \hat{a}^\dagger) U(\eta) = e^{-2\eta}(\hat{a} - \hat{a}^\dagger) \]  \hspace{1cm} (1)

b) (3 pts.) Calculate \( \Delta x \) and \( \Delta p \) for the state \( |\eta\rangle = U(\eta)|0\rangle \). Is this a minimum uncertainty state?

c) (3 pts.) Verify Eq. (1). (Hint: Differentiate both sides with respect to \( \eta \).)

**Problem 4**

Consider a dumbbell constrained to rotate in the \( x - y \) plane about its center of mass.

The Hamiltonian is given by
\[ \hat{H} = \frac{\hat{I}_z^2}{2I}, \]
where \( I \) is the moment of inertia of the dumbbell.

a) (3 pts.) What are the allowed energy eigenvalues and what is their degeneracy? Suppose the dumbbell’s wavefunction at time \( t = 0 \) is given by
\[ \psi(\theta, t = 0) = \frac{1}{\sqrt{2\pi}} \left( \sqrt{\frac{1}{3}}e^{i\theta} + i\sqrt{\frac{2}{3}}e^{2i\theta} \right). \]

b) (2 pts.) What is \( \langle E \rangle \)?

c) (5 pts.) What is the probability density of finding the dumbbell at angle \( \theta \) at time \( t \), (i.e. what is \( |\psi(\theta, t)|^2 \))?