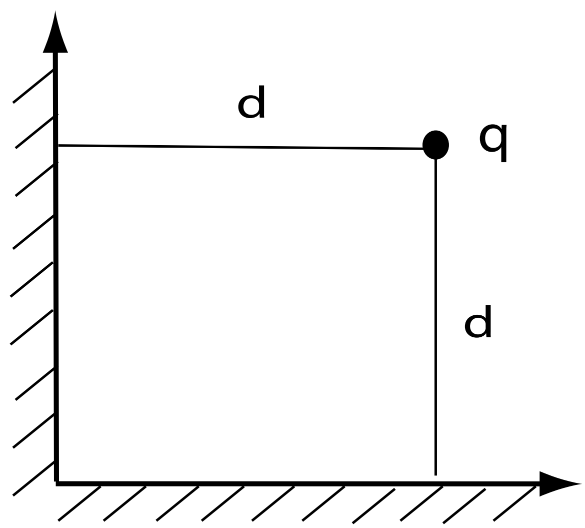
Final Exam
Phy 182 - Fall 2007
Wednesday, Dec. 12, 3-5 pm

This exam contains 6 short answer questions and 3 problems. Useful equations are collected at the end of the exam. Please sign a pledge on your exam stating that you have abided by Duke's honor code.

Short Answer 1 (10 pts.)

Consider a point charge, q, located near a corner made by two semi-infinite perfect conducting plates at right angles, as depicted in the figure. The distance of the charge from each conducting plate is d.

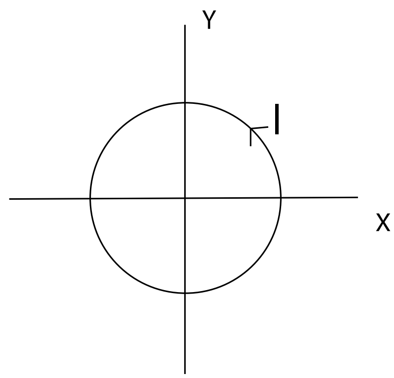


a) Solve for the electrostatic potential using the method of images. Indicate the locations and charges of each image charge. Show that the potential obtained from the image charges satisfies the required boundary conditions.

b) Calculate the force on the charge q.

Short Answer 2 (10 pts.)

Consider a circular loop of radius d carrying current I. The loop is in the x-y plane with its center at the origin. Write down an expression for the $\vec{B}$ field far from the loop, $r \gg d$, keeping the lowest nontrivial term in the multipole expansion of the $\vec{B}$ field.



Short Answer 3 (10 pts.)

Suppose magnetic charge exists in nature so that the Maxwell equation for $\vec{B}$ becomes

$$\vec{\nabla} \cdot \vec{B} = \mu_0 \, \rho_m \, .$$

Furthermore this charge should be conserved *locally* so that

$$\frac{\partial}{\partial t} \rho_m + \vec{\nabla} \cdot \vec{J}_m = 0 \, ,$$

where $\vec{J}_m$ is the magnetic current. How should Faraday's law be modified

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = ?? \, ,$$

so that a consistent set of equations is obtained? Should the other Maxwell equations be modified?

Short Answer 4 (10 pts.)

Write down the boundary conditions that relate static $\vec{E}$ and $\vec{B}$ fields on either side of a two-dimensional surface carrying surface charge density, σ, and surface current density, $\vec{K}$. You may assume that the volume on either side of surface is a vacuum. (No dielectrics or permeable media).

Short Answer 5 (10 pts.)

a) Write down the fields $\vec{E}$ and $\vec{B}$ in terms of the electrostatic potential, V, and magnetic vector potential, $\vec{A}$.

b) Explain the gauge invariance of the Maxwell equations. How do V and $\vec{A}$ transform under gauge transformations? Show that $\vec{E}$ and $\vec{B}$ are invariant under gauge transformations.

c) Which of the Maxwell equations can be solved simply by writing $\vec{E}$ and $\vec{B}$ in terms of gauge potentials? Demonstrate how this works.

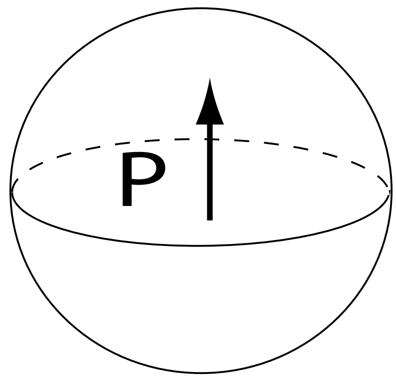
Short Answer 6 (10 pts.)

a) Starting from the Maxwell equations, **derive** wave equations for $\vec{E}$ and $\vec{B}$ in vacuum. Show that they admit plane wave solutions and find the relationship between the angular frequency, ω, and wavenumber, k.

b) How are the results modified in the presence of a linear material with permittivity ϵ and permeability μ. What is the phase velocity of the wave in the medium?

Problem 1 (20 pts.)

A sphere of radius R contains a material with uniform polarization, $\vec{P} = P\hat{z}$.



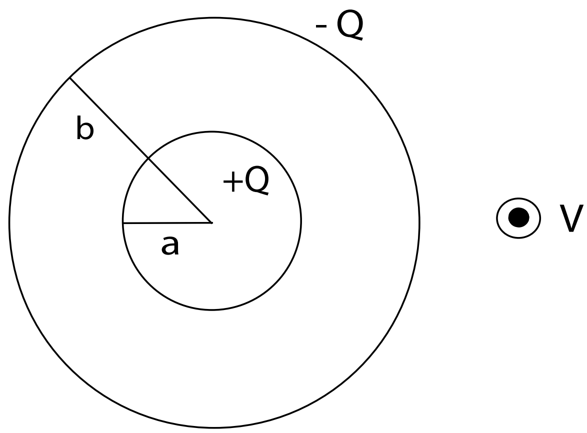
a) Compute the dipole moment of the sphere.

a) Compute bound and surface charge distributions.

c) Compute the electrostatic potential inside and outside the sphere. Check that your field the outside the sphere is consistent with your answer to part a).

Problem 2 (20 pts.)

The figure below depicts a cross sectional view of two infinite concentric cylinders. The interior cylinder, of radius a, carries charge $+Q$ per unit length, and the outer cylinder, of radius b, carries charge $-Q$ per unit length. The cylinders are moving out of the page with velocity v.



a) Compute $\vec{E}$ and $\vec{B}$ for this configuration. Make sure you clearly state what $\vec{E}$ and $\vec{B}$ are in each of three regions: i) $r \leq a$, ii) $a < r < b$, and iii) $r \geq b$.

b) Compute the angular momentum per unit length carried by the fields.

c) Suppose that the cylinders are brought to a stop. $\vec{B}$ must go from the value you answered in part a) to zero. What is the induced emf, in terms of $\partial\vec{B}/\partial t$? What is the torque exerted on each of the cylinders by the emf? Show that the total angular momentum imparted to the cylinders by the induced emf is equal to the original angular momentum carried by the fields.

Problem 3 (20 pts.)

a) Let (x, y, z, t) be the coordinates used by an observer at rest, and (x', y', z', t') be the coordinates used by another observer moving with velocity v in the x-direction. Write down the Lorentz transformation that relates these two coordinate systems. Check that the transformation satisfies $\vec{x}^{\,2} - c^2\, t^2 = \vec{x}'^{\,2} - c^2\, t'^{\,2}$.

b) Consider an infinite line of charge along the x-axis. In the rest frame of part a), the line charge is static with uniform charge density, λ. What are the $\vec{E}$ and $\vec{B}$ fields observed by the moving observer of part a)?

c) A particle of mass m_A decays into two particles with masses m_B and m_C? What is the energy of the particle with mass m_B (in the rest frame of the decaying particle)?

Formulae Sheet

Maxwell Equations in Vacuum

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

Maxwell Equations in Matter

$$\vec{\nabla} \cdot \vec{D} = \rho_f \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}_f$$

Displacement Field and $\vec{H}$

$$\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P} \qquad \vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

Linear Media

$$\vec{D} = \epsilon \, \vec{E} \qquad \vec{H} = \frac{1}{\mu} \vec{B}$$

Lorentz Force Law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Conductors

$$\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \qquad V = I\,R \qquad P = V\,I$$

Inductance

$$\Phi_1 = M_{12}\,\Phi_2 \qquad \Phi = \oint d\vec{a} \cdot \vec{B} = L\,I$$

Electromagnetic Field Energy

$$U = \frac{1}{2} \int d^3x \left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0} \vec{B}^2 \right)$$

Poynting Vector, Momentum

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) \qquad \vec{P} = \int d^3x \, \epsilon_0 (\vec{E} \times \vec{B})$$

Solution to Laplace's Equation in Spherical Coordinates

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l \, r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{3x^2 - 1}{2} \qquad \int_0^1 dx \, P_l(x) P_m(x) = \frac{2}{2l+1} \delta_{lm}$$