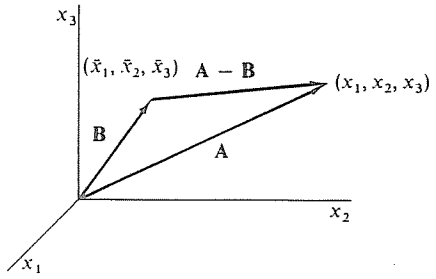

Figure 1-13

defined by the vector B is

$$\sqrt{\sum_i (x_i - \bar{x}_i)^2} = \sqrt{(\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B})} = |\mathbf{A} - \mathbf{B}|$$

That is, we can define the vector connecting any point with any other point as the difference of the position vectors that define the individual points, as in Figure 1-13. The distance between the points is then the magnitude of the difference vector. And since this magnitude is the square root of a scalar product, it is invariant to a coordinate transformation. This is an important fact and can be summarized by the statement that *orthogonal transformations are distance-preserving transformations.* Also, the angle between two vectors is preserved under an orthogonal transformation. These two results are essential if we are successfully to apply transformation theory to physical situations.

The scalar product obeys the commutative and distributive laws:

$$\mathbf{A} \cdot \mathbf{B} = \sum_i A_i B_i = \sum_i B_i A_i = \mathbf{B} \cdot \mathbf{A} \tag{1.59}$$

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \sum_i A_i (B + C)_i = \sum_i A_i (B_i + C_i) \\ &= \sum_i (A_i B_i + A_i C_i) = (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{C}) \end{aligned} \tag{1.60}$$

1.11 Unit Vectors

Sometimes we want to describe a vector in terms of the components along the three coordinate axes together with a convenient specification of these axes. For this purpose, we introduce the **unit vectors** along the rectangular axes* $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The following ways of expressing the vector A are equivalent:

$$\mathbf{A} = (A_1, A_2, A_3) \quad \text{or} \quad \mathbf{A} = \mathbf{e}_1 A_1 + \mathbf{e}_2 A_2 + \mathbf{e}_3 A_3 = \sum_i \mathbf{e}_i A_i \tag{1.61}$$

We obtain the components of the vector A by projection onto the axes:

$$A_i = \mathbf{e}_i \cdot \mathbf{A} \tag{1.62}$$

*There are many variants of the symbols for unit vectors. Perhaps the most common set is **i**, **j**, **k** or $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$; others include $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ and I_x, I_y, I_z.

We have seen (Equation 1.56) that the scalar product of two vectors has a magnitude equal to the product of the individual magnitudes multiplied by the cosine of the angle between the vectors:

$$\mathbf{A} \cdot \mathbf{B} = AB \cos(\mathbf{A}, \mathbf{B}) \tag{1.63}$$

Therefore the scalar product of any two unit vectors (because they are orthogonal) becomes

$$\boxed{\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}} \tag{1.64}$$

EXAMPLE 1.4

Two position vectors are expressed in Cartesian coordinates as $\mathbf{A} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. Find the magnitude of the vector from point A to point B, the angle θ between A and B, and the component of B in the direction of A.

Solution: The vector from point A to point B is $\mathbf{B} - \mathbf{A}$ (see Figure 1-13).

$$\mathbf{B} - \mathbf{A} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} - (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 3\mathbf{i} - \mathbf{k}$$

$$|\mathbf{B} - \mathbf{A}| = \sqrt{9 + 1} = \sqrt{10}$$

From Equation 1.56

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \frac{(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \cdot (4\mathbf{i} + 2\mathbf{i} - 3\mathbf{k})}{\sqrt{9}\sqrt{29}}$$

$$\cos\theta = \frac{4 + 4 + 6}{3(\sqrt{29})} = 0.867$$

$$\theta = 30°$$

The component of B in the direction of A is $B \cos\theta$ and, from Equation 1.56,

$$B\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{A} = \frac{14}{3} = 4.67$$

1.12 The Vector Product of Two Vectors

We next consider another method for combining two vectors—the so-called **vector product** (or **cross product**). First, we assert that this operation does, in fact, produce a vector.* The vector product of A and B is denoted† by a bold cross $\times$,

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{1.65}$$

*The product actually produces an *axial* vector, but the term *vector product* is used to be consistent with popular usage. [See Marion (Ma65a, Section 1.15).]

†Older notations include [**AB**], [**A** . **B**], and [**A** ∧ **B**].

where $\mathbf{C}$ is the vector resulting from this operation. The components of $\mathbf{C}$ are defined by the relation

$$\boxed{C_i \equiv \sum_{j,k} \varepsilon_{ijk} A_j B_k} \tag{1.66}$$

where the symbol ε_{ijk} is the **permutation symbol** (or **Levi-Civita density**) and has the following properties:

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{if any index is equal to any other index} \\ +1, & \text{if } i, j, k \text{ form an } \textit{even} \text{ permutation of } 1, 2, 3 \\ -1, & \text{if } i, j, k \text{ form an } \textit{odd} \text{ permutation of } 1, 2, 3 \end{cases} \tag{1.67}$$

Thus

$$\varepsilon_{122} = \varepsilon_{313} = \varepsilon_{211} = 0, \quad \text{etc.}$$
$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$$
$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

Using the preceding notation, the components of $\mathbf{C}$ can be explicitly evaluated. For $i = 1$, the only nonvanishing ε_{ijk} are ε_{123} and ε_{132}—that is, for $j, k = 2, 3$ in either order. Therefore

$$C_1 = \sum_{j,k} \varepsilon_{1jk} A_j B_k = \varepsilon_{123} A_2 B_3 + \varepsilon_{132} A_3 B_2$$
$$= A_2 B_3 - A_3 B_2 \tag{1.68a}$$

Similarly,

$$C_2 = A_3 B_1 - A_1 B_3 \tag{1.68b}$$
$$C_3 = A_1 B_2 - A_2 B_1 \tag{1.68c}$$

Consider now the expansion of the quantity $[AB\sin(\mathbf{A}, \mathbf{B})]^2 = (AB\sin\theta)^2$:

$$A^2B^2\sin^2\theta = A^2B^2 - A^2B^2\cos^2\theta$$
$$= \left(\sum_i A_i^2\right)\left(\sum_i B_i^2\right) - \left(\sum_i A_i B_i\right)^2$$
$$= (A_2B_3 - A_3B_2)^2 + (A_3B_1 - A_1B_3)^2 + (A_1B_2 - A_2B_1)^2 \tag{1.69}$$

where the last equality requires some algebra. Identifying the components of $\mathbf{C}$ in the last expression, we can write

$$(AB\sin\theta)^2 = C_1^2 + C_2^2 + C_3^2 = |\mathbf{C}^2| = C^2 \tag{1.70}$$

If we take the positive square root of both sides of this equation,

$$C = AB\sin\theta \tag{1.71}$$

This equation states that if $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, then the magnitude of $\mathbf{C}$ is equal to the product of the magnitudes of $\mathbf{A}$ and $\mathbf{B}$ multiplied by the sine of the angle between them. Geometrically, $AB\sin\theta$ is the area of the parallelogram defined by the vectors $\mathbf{A}$ and $\mathbf{B}$ and the angle between them, as in Figure 1-14.

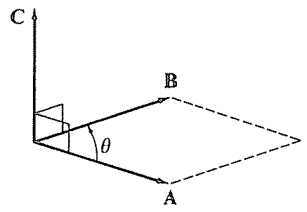

Figure 1-14

EXAMPLE 1.5

Show by using Equations 1.52 and 1.66 that

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{1.72}$$

Solution: Using Equation 1.66, we have

$$(\mathbf{B} \times \mathbf{C})_i = \sum_{j,k} \varepsilon_{ijk} B_j C_k$$

Using Equation 1.52, we have

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \sum_{i,j,k} \varepsilon_{ijk} A_i B_j C_k \tag{1.73}$$

Similarly, for the right-hand side of Equation 1.72 we have

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \sum_{i,j,k} \varepsilon_{ijk} C_i A_j B_k$$

From the definition (Equation 1.67) of ε_{ijk} we can interchange two adjacent indices of ε_{ijk}, which changes the sign.

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \sum_{i,j,k} -\varepsilon_{jik} C_i A_j B_k$$
$$= \sum_{i,j,k} \varepsilon_{jki} A_j B_k C_i \tag{1.74}$$

Because the indices i, j, k are dummy and can be renamed, the right-hand sides of Equations 1.73 and 1.74 are identical, and Equation 1.72 is proved. Equation 1.72 can also be written as $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$, indicating that the scalar and vector products can be interchanged as long as the vectors stay in the order $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Notice that, if we let $\mathbf{B} = \mathbf{A}$, we have

$$\mathbf{A} \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{A}) = 0$$

showing that $\mathbf{A} \times \mathbf{C}$ must be perpendicular to $\mathbf{A}$.

$\mathbf{A} \times \mathbf{B}$ (i.e., $\mathbf{C}$) is perpendicular to the plane defined by $\mathbf{A}$ and $\mathbf{B}$ because $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0$ and $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{B}) = 0$. Since a plane area can be represented by a

vector normal to the plane and of magnitude equal to the area, C is evidently such a vector. The positive direction of C is chosen to be the direction of advance of a right-hand screw when rotated from A to B.

The definition of the vector product is now complete; components, magnitude, and geometrical interpretation have been given. We may therefore reasonably expect that C is indeed a vector. The ultimate test, however, is to examine the transformation properties of C. This is carried out in Marion (Ma65a, Section 1.14), where it is shown that C does in fact transform as a vector under a proper rotation.

We should note the following properties of the vector product that result from the definitions:

$$\text{(a)} \quad \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1.75}$$

but, in general,

$$\text{(b)} \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \tag{1.76}$$

Another important result (see Problem 1-22) is

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{1.77}$$

EXAMPLE 1.6

Find the product of $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$.

Solution:

$$(\mathbf{A} \times \mathbf{B})_i = \sum_{j,k} \varepsilon_{ijk} A_j B_k$$

$$(\mathbf{C} \times \mathbf{D})_i = \sum_{l,m} \varepsilon_{ilm} C_l D_m$$

The scalar product is then computed according to Equation 1.52:

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \sum_i \left(\sum_{j,k} \varepsilon_{ijk} A_j B_k \right) \left(\sum_{l,m} \varepsilon_{ilm} C_l D_m \right)$$

Rearranging the summations, we have

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \sum_{\substack{l,m \\ j,k}} \left(\sum_i \varepsilon_{jki} \varepsilon_{lmi} \right) A_j B_k C_l D_m$$

where the indices of the ε's have been permuted (twice each so that no sign change occurs) to place in the third position the index over which the sum is carried out. We can now use an important property of the ε_{ijk} (see Problem 1-22):

$$\boxed{\sum_k \varepsilon_{ijk} \varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}} \tag{1.78}$$

We therefore have

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \sum_{\substack{j,k \\ l,m}} (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) A_j B_k C_l D_m$$

Carrying out the summations over j and k, the Kronecker deltas reduce the expression to

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \sum_{l,m} (A_l B_m C_l D_m - A_m B_l C_l D_m)$$

This equation can be rearranged to obtain

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \left(\sum_l A_l C_l \right) \left(\sum_m B_m D_m \right) - \left(\sum_l B_l C_l \right) \left(\sum_m A_m D_m \right)$$

Since each term in parentheses on the right-hand side is just a scalar product, we have, finally,

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D})$$

The orthogonality of the unit vectors $\mathbf{e}_i$ requires the vector product to be

$$\mathbf{e}_i \times \mathbf{e}_j = \mathbf{e}_k, \quad i, j, k \text{ in cyclic order} \tag{1.79a}$$

We can now use the permutation symbol to express this result as

$$\boxed{\mathbf{e}_i \times \mathbf{e}_j = \mathbf{e}_k \varepsilon_{ijk}} \tag{1.79b}$$

The vector product $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, for example, can now be expressed as

$$\boxed{\mathbf{C} = \sum_{i,j,k} \varepsilon_{ijk} \mathbf{e}_i A_j B_k} \tag{1.80a}$$

By direct expansion and comparison with Equation 1.80a, we can verify a determinantal expression for the vector product:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{1.80b}$$

We state the following identities without proof:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \equiv \mathbf{ABC} \tag{1.81}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{1.82}$$

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= \mathbf{A} \cdot [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] \\ &= \mathbf{A} \cdot [(\mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{B} \cdot \mathbf{C})\mathbf{D}] \\ &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \end{aligned} \tag{1.83}$$

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) &= [(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{D}]\mathbf{C} - [(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}]\mathbf{D} \\ &= (\mathbf{ABD})\mathbf{C} - (\mathbf{ABC})\mathbf{D} \end{aligned} \tag{1.84}$$