

Physics 53

Systems of Particles

Everything should be as simple as it is, but not simpler.

— Albert Einstein

Overview

An object of ordinary size — which we call a “macroscopic” system — contains a huge number of atoms or molecules. It is out of the question to attempt to use the laws we have discussed for a single particle to describe separately each particle in such a system.

There are nevertheless some relatively simple aspects of the behavior of a macroscopic system that (as we will show) follow from the basic laws for a particle. We will be able to show that in any multi-particle system each of the mechanical quantities relevant to the state of the system consists of two parts:

1. One part in which the system is treated as though it were a single particle (with the total mass of the system) located at a special point called the **center of mass**. This part is often called the *CM motion*.
2. Another part describing the *internal motion* of the system, as seen by an observer located at (and moving with) the center of mass.

In this section we will give a general analysis of the variables to be used in describing the motion of *any* system of particles, large or small. In later sections we will apply this analysis to the case of solid objects, in the approximation that all the particles in the object have a fixed spatial relation to each other. In this approximation, the object is called a “rigid body.”

Still later we will discuss fluids, including gases, where things are more complicated because the internal motion is dominant, and where we must take a statistical approach in the analysis.

Center of mass

We consider a system composed of N point particles, each labeled by a value of the index i which runs from 1 to N . Each particle has its own mass m_i and (at a particular time) is located at its particular place \mathbf{r}_i .

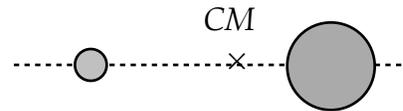
The **center of mass (CM)** of the system is defined by the following position vector:

Center of mass	$\mathbf{r}_{CM} = \frac{1}{M} \sum_i m_i \mathbf{r}_i$
----------------	---

where M is the total mass of all the particles.

This vector locates a point in space — which may or may not be the position of any of the particles. It is the *mass-weighted average* position of the particles, being nearer to the more massive particles.

As a simple example, consider a system of only two particles, of masses m and $2m$, separated by a distance ℓ . Choose the coordinate system so that the less massive particle is at the origin and the other is at $x = \ell$, as shown in the drawing. Then we have $m_1 = m$, $m_2 = 2m$,



$x_1 = 0$, $x_2 = \ell$. (The y and z coordinates are zero of course.) We find from the definition $x_{CM} = \frac{1}{3m}(m \cdot 0 + 2m \cdot \ell) = \frac{2}{3}\ell$. The location is indicated on the drawing.

As time goes on the position vectors of the particles \mathbf{r}_i generally change with time, so in general the CM moves. The velocity of its motion is the time derivative of its position:

$$\mathbf{v}_{CM} = \frac{1}{M} \sum_{i=1}^N m_i \mathbf{v}_i.$$

But the sum on the right side is just the *total* (linear) momentum of all the particles. Solving for this, we find an important result:

Total momentum of a system	$\mathbf{p}_{tot} = M\mathbf{v}_{CM}$
----------------------------	---------------------------------------

This is just like the formula for the momentum of a single particle, so we see that:

The total momentum of a system is the same as if all the particles were located together at the CM and moving with its velocity.

Total force

The total force on the i th particle consists of the (net) *external* force on it, plus the net force due to the interactions with other particles in the system, which we call the *internal* forces. We write this out as follows:

$$\mathbf{F}_i = \mathbf{F}_i^{\text{ext}} + \sum_j \mathbf{F}_{i/j},$$

where $\mathbf{F}_{i/j}$ denotes the force exerted *on* the i th particle *by* the j th particle.

To get the total force on the whole system, we simply add up all these forces:

$$\mathbf{F}_{\text{tot}} = \sum_i \mathbf{F}_i^{\text{ext}} + \sum_i \sum_{j \neq i} \mathbf{F}_{i/j}.$$

In the double sum on the right the terms cancel in pairs by Newton's 3rd law (for example, $\mathbf{F}_{1/2} + \mathbf{F}_{2/1} = 0$). The double sum thus gives zero, so

$$\mathbf{F}_{\text{tot}} = \sum_i \mathbf{F}_i^{\text{ext}}.$$

The total force on the system is the sum of only the external forces.

Since there are very many internal forces, the fact that they give no *net* contribution to the total force is why it is possible for many applications to treat an object of macroscopic size as a single particle. The internal forces do play important roles in determining some aspects of the system, such as its energy.

For each particle individually we have Newton's 2nd law:

$$\mathbf{F}_i = \frac{d\mathbf{p}_i}{dt}.$$

Adding these for all the particles, and using the above result for \mathbf{p}_{tot} , we find two forms of the 2nd law as it applies to systems of particles:

Newton's 2nd law for systems	$\mathbf{F}_{\text{tot}}^{\text{ext}} = \frac{d\mathbf{p}_{\text{tot}}}{dt} = M\mathbf{a}_{\text{cm}}$
------------------------------	--

Here $\mathbf{a}_{\text{CM}} = d\mathbf{v}_{\text{CM}} / dt$ is the *acceleration* of the CM. We see that:

The total external force produces an acceleration of the center of mass, as though all the particles were located there.

This is not the only possible effect of the external forces. They can also cause rotational motion *about* the CM, as we will see later. But the *internal* forces do not change the motion of the CM.

From the first formula above we find one of the most important laws of mechanics:

Conservation of momentum	If the total external force on a system is zero, the total momentum of the system is conserved.
--------------------------	---

In physics, the term “is conserved” means “remains constant in time.”

Two important points about this conservation law:

- Only *external* forces are at issue here. There can be (and often are) many internal forces of great complexity acting on the individual particles and making their individual motions very complicated. But only *external* forces can change the *total* momentum of the system.
- Since force and momentum are both vectors, this conservation law holds *separately* for each component. Thus if the x component of the total external force is zero, the x component of the total momentum is conserved, regardless of whether the other components are also conserved.

CM reference frame

We have found that in some important respects a system of particles behaves as though all the particles were located at the CM. But the particles are not in fact all at the CM, so some aspects of the motion must be more complicated.

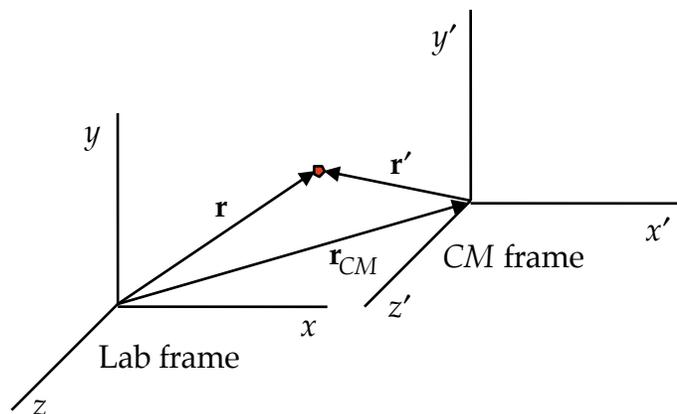
To analyze the situation further, it is useful to introduce a coordinate system with the CM as its origin. This defines the **CM reference frame** which may or may not be an inertial frame. We will call the (inertial) frame of our previous analysis the “lab” frame.

Shown are the two frames, with the position of the i th particle relative to each frame. The vector triangle shows that the two position vectors are related by

$$\mathbf{r}'_i = \mathbf{r}_i - \mathbf{r}_{CM}$$

The velocities (time derivatives of the positions) are thus related by

$$\mathbf{v}'_i = \mathbf{v}_i - \mathbf{v}_{CM}$$



From this we can calculate the total momentum in the CM frame:

$$\sum_i m_i \mathbf{v}'_i = \sum_i m_i \mathbf{v}_i - M \mathbf{v}_{CM}.$$

As we saw earlier, the two terms on the right are equal and therefore cancel. So we have

The total momentum, as measured in the CM frame, is zero.

In some applications this property is taken to be the *definition* of the CM frame.

Kinetic energy of a system

In the lab frame, the total kinetic energy of all the particles is

$$\begin{aligned} \frac{1}{2} \sum_i m_i v_i^2 &= \frac{1}{2} \sum_i m_i (\mathbf{v}'_i + \mathbf{v}_{CM})^2 \\ &= \frac{1}{2} \sum_i m_i \left[v_i'^2 + v_{CM}^2 + 2 \mathbf{v}'_i \cdot \mathbf{v}_{CM} \right] \\ &= \frac{1}{2} M v_{CM}^2 + \frac{1}{2} \sum_i m_i v_i'^2 + \mathbf{v}_{CM} \cdot \sum_i m_i \mathbf{v}'_i \end{aligned}$$

The sum in the very last term is zero, as we have just seen, so we have

$$\frac{1}{2} \sum_i m_i v_i^2 = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} \sum_i m_i v_i'^2.$$

The two terms on the right have simple interpretations:

- The first term is what the kinetic energy would be if all the particles really were at the CM and moving with its speed. We often call this the kinetic energy *of* the CM motion.
- The second term is the total kinetic energy as it would be measured by an observer in the CM reference frame. We call this the kinetic energy *relative to* the CM, or sometimes the *internal* kinetic energy.

The total kinetic energy (in the lab frame) is the sum of these two terms:

Kinetic energy of a system	$K = \frac{1}{2} M v_{CM}^2 + K(\text{rel. to CM})$
----------------------------	---

This breakup of the kinetic energy into that of the CM plus that relative to the CM is an example of the general property stated on page 1. We will see later that this property also holds for angular momentum.

It holds for linear momentum too, but the second part, the total linear momentum relative to the CM, is always zero.

Impulse and average force

The 2nd Law says that the total (external) force is equal to the rate of change of total (linear) momentum. It follows that

$$d\mathbf{p}_{tot} = \mathbf{F}_{tot} dt.$$

Integrating both sides we find for the net change in momentum

$$\Delta\mathbf{p}_{tot} = \int_{t_0}^{t_1} \mathbf{F}_{tot} dt.$$

This integral (over time) of the force is called the **impulse**. We have shown a theorem:

Impulse-momentum theorem	The impulse of the total force is equal to the change of the total linear momentum.
--------------------------	---

One use of this relation is to define the *average* force that acts during a specified time interval. Let the force act for time Δt , producing a net change $\Delta\mathbf{p}$ in the total momentum. The average force is given by

Average force	$\mathbf{F}_{av} = \frac{\Delta\mathbf{p}_{tot}}{\Delta t}$
---------------	---

This is useful in cases where the force is an unknown function of time and we would like to describe its average effect over some specific time interval without having to investigate the detailed behavior.

Two body collisions

There are numerous situations where two objects interact through a force that has effect only when the objects are close to each other. This is called a “short-range” force.

The normal force between objects in “contact” is an example. In such cases “close” or “in contact” mean that the distances between atoms on the adjacent surfaces of the two objects are comparable to the diameter of a single atom (about 10^{-10} m).

If two objects which are originally well separated move toward each other without interacting until a short-range force comes into effect, we have a “collision”. There are two important cases, differing by the nature of the short-range forces:

- The short-range forces between atoms at the surfaces of the objects are strongly repulsive, so the two objects move away from each other after the very short time during which the forces are effective.
- Those forces are strongly attractive, causing the objects to stick together. After a very short time the CM velocities of the two objects become the same, and they move off together (or come to rest, if the total momentum is zero).

In either case, the “collision time” during which the states of the particles of the system change from initial states to final states is very short, typically milliseconds.

Let the two objects *taken together* constitute the system under consideration. Then the short-range interaction between them (the “collision force”, about which we are likely to know very little) is an *internal* force and therefore does not affect the total momentum. If there are no *external* forces, then *total momentum is conserved*.

This is a great simplification. The collision forces can be quite complicated, and indeed the internal forces within each body may come into play in important ways if the behavior of each object separately is to be analyzed. But if the objects are treated as parts of a *single* system, then total momentum is conserved if external forces are absent or negligible.

Even if there are external forces (such as gravity), the collision time may be so short that the momentum change produced by external forces during that time can be neglected. We can then reasonably assume that the system's total momentum is approximately conserved during the brief time of the collision — although it will be changed by the external forces over longer periods of time. This situation often occurs in practice.

In classical physics one also assumes that the total mass of the system does not change in the collision, although some mass may be transferred from one object to the other.

Remarkably, conservation of total mass is only a “classical” approximation, valid when all speeds are small compared to the speed of light. Special relativity shows that mass (or rather “rest energy” mc^2) is *not* generally conserved, and indeed can be interchanged with other forms of energy.

What about conservation of energy in a collision? We distinguish two cases:

- First, suppose the collision force is *conservative*. During the brief collision time the objects have moved only a negligible distance, so the potential energy associated with the collision force has not changed appreciably, and external forces have done only negligible work. Thus for conservative collision forces the total *kinetic energy* is conserved during the collision. Such a collision is called **elastic**.
- If the collision force is *not* conservative, then in general kinetic energy is not conserved. In that case the collision is called **inelastic**. Cases in which the objects stick together after the collision are sometimes called “totally” inelastic.

For macroscopic objects there are always some non-conservative aspects of the collision, but they may be negligible in practice.

The rules for comparing the situations *just before and after* a collision are these:

Two-body collisions

Total momentum is conserved.
Total mass is conserved.
If the collision is elastic, total kinetic energy is conserved.