Exam II

Solutions

Part A: Multiple choice questions. Check the best answer. Each question carries a value of 4 points.

1. A system is subject to only two external forces, of equal magnitude.
   - If the forces are in opposite directions but not along the same line, there is at least one point about which total angular momentum is conserved. [Total torque is \((\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}\), independent of the reference point and never zero.]
   - √ If the forces are in opposite directions and along the same line, total angular momentum is conserved about every point. [Now \(\mathbf{r}_2 - \mathbf{r}_1\) is parallel to \(\mathbf{F}\).]
   - □ If the forces are in the same direction but not along the same line, there is no point about which total angular momentum is conserved. [About the point halfway between them the total torque is zero.]
   - □ If the forces are in the same direction and along the same line, there is no point about which total angular momentum is conserved. [About any point on that line the total torque is zero.]

2. You are at one end of a platform on wheels that can roll freely on a horizontal floor. You and the platform are at rest. The platform has length \(\ell\), and you and the platform have the same mass \(m\). You walk to the other end of the platform and stop. Which of the following is NOT true?
   - □ The CM of the system of you plus platform does not move.
   - □ The platform moves across the floor distance \(\ell/2\) during your walk.
   - □ You do not move at all relative to the floor. [You move distance \(\ell/2\) the other way.]
   - √ Total horizontal momentum of the system is conserved.

3. The door shown has two hinges supporting it. Over time:
   - □ The screws holding the bottom hinge to the door and the wall tend to work loose, because the door pulls on the hinge.
   - √ The screws holding the top hinge to the door and the wall tend to work loose, because the door pulls on the hinge.
   - □ Both sets of screws tend to pull loose, because the door pulls on both hinges.
   - □ There is no force pulling either set of screws loose. [Consider torques about the bottom hinge.]
4. A satellite is in an elliptical orbit around the earth. Points $A$ and $B$ are at opposite ends of the major axis of the ellipse, and the distance between them is $2a$.

- [ ] The earth is at the center of the ellipse.
- [ ] This orbit has less energy than a circular orbit of radius $a$.
- [ ] The speed of the satellite is the same at $A$ and $B$.
- [✓] None of the above is true.

Part B: Check True or False. Each question carries a value of 3 points.

1. The melting of the glaciers that covered much of the northern parts of the earth, allowing the water to flow into the oceans, resulted in a slight lengthening of the day.

   - [✓] True [The moment of inertia increased, so $\omega$ decreased.]
   - [ ] False

2. To round a curve on a level road, a bicyclist leans his bike so that the line from the CM to the point of contact between tires and road is along $g_{\text{eff}}$.

   - [✓] True
   - [ ] False

3. Tides are especially large when the sun and moon are both on one side of the earth, but especially small when they are on opposite sides.

   - [ ] True
   - [✓] False [Large in both cases.]
Part C: Problems. Work problems in the space provided, indicating your method clearly. A correct final answer supported by no argument, or a fallacious one, will receive little or no credit. The problems carry the point values shown.

1. A small ball of mass $m$ collides elastically with the flat surface of a large block of mass $M$ as shown. Its initial momentum $\mathbf{p}$ makes angle $\theta$ with the normal to the block’s surface, and its final momentum $\mathbf{p}'$ makes angle $\theta'$ with the normal. The surface the ball strikes is frictionless so the block can exert no force on the ball parallel to the surface. After the collision the block moves with momentum $\mathbf{P}$.

   a. What is the direction of $\mathbf{P}$? How do you know?

   b. Write the equations for the conservation laws in terms of the variables given, using the coordinate system shown. [Use $p^2 / 2m$ for kinetic energy of a particle.]

   c. Show that if $M \to \infty$, then $p' = p$, $\theta' = \theta$, and $P = 2p \cos \theta$.

   [15 points]

   a. In the $+x$-direction (to the right). If the block can exert no force in the $y$-direction on the ball, then the ball can exert no force in the $y$-direction on the block (3rd law).

   b. Cons. of $P_x$: $p \cos \theta = P - p' \cos \theta'$. Cons. of $P_y$: $p \sin \theta = p' \sin \theta'$. Cons. of $K$: $p^2 / 2m = p'^2 / 2m + P^2 / 2M$.

   c. As $M \to \infty$ we see from conservation of $K$ that $p' = p$. Then from conservation of $P_y$ we see $\theta' = \theta$, and from conservation of $P_x$ that $P = 2p \cos \theta$. The block absorbs momentum but negligible kinetic energy.
2. A wheel of mass $m$ and radius $R$ is to be lifted over a curb by a force $F$ exerted at its axle as shown. The line from the corner of the curb to the axle makes angle $\theta$ with the direction of $F$ and angle $\alpha$ with the vertical. Although $\alpha$ is fixed by the geometry, we can vary $\theta$.
   a. If the wheel is just on the verge of rising, what is $F$ in terms of $m$, $g$ and the angles?
   b. For what value of $\theta$ is $F$ the smallest, and what is that value of $F$?

   [Consider torques about the corner of the curb.]

   [10 points]

   a. If the wheel is on the verge of rising, the normal force from the street is zero. The only forces giving torques about the corner of the curb are $F$ and the weight. For $F$ the moment arm is $R\sin\theta$, and the torque is clockwise. For the weight the moment arm is $R\sin\alpha$ and the torque is counterclockwise. Thus we have $FR\sin\theta = mgR\sin\alpha$, or $F = mg\sin\alpha / \sin\theta$.

   b. This is obviously smallest when $\sin\theta$ is largest, or when $\theta = 90^\circ$ and $F = mg\sin\alpha$. 
3. The object shown hangs at rest on a frictionless pivot. A small ball of mass $m$ moving horizontally as shown with speed $v_0$ collides with the object at a distance $x$ below the pivot. The object has mass $M$, moment of inertia $I$ about the pivot, and its CM is at distance $x_{CM}$ below the pivot. Just after the collision the ball continues to move horizontally with velocity $v$.

a. Find the angular speed $\omega$ of rotation of the object about the pivot just after the collision, in terms of the given quantities. [What is conserved, regardless of the nature of the collision?]

b. Express the total horizontal momentum of the system just after the collision, in terms of the given quantities only. [What is the speed of the CM of the object in terms of $x_{CM}$ and $\omega$?]

c. Find the value of $x$, in terms of $M$, $I$ and $x_{CM}$, for which horizontal momentum is conserved, so that the pivot exerts no horizontal force.

[15 points]

| a. Angular momentum about the pivot is conserved. We have $m v_0 x = m v x + I \omega$, or $\omega = \frac{m x}{I} (v_0 - v)$. |
| b. The CM speed of the object is $v_{CM} = x_{CM} \cdot \omega$. The final total horizontal momentum is $P_f = m v + M v_{CM} = m v + M x_{CM} \cdot \omega$. Using the result of (a) we find $P_f = m v + \frac{M x_{CM} x}{I} m (v_0 - v)$. |
| c. Setting the initial and final momenta equal we have $m v_0 = m v + \frac{M x_{CM} x}{I} m (v_0 - v)$, or $1 = \frac{M x_{CM} x}{I}$. This gives $x = \frac{I}{M x_{CM}}$. [This point is called the “center of percussion” or, in sports, the “sweet spot”.] |
4. A billiard ball of mass $m$, radius $R$, moment of inertia about its CM $I = \frac{2}{5}mR^2$ is at rest on a horizontal table. It is struck a sharp horizontal blow by a cue stick, at height $h$ above the table as shown. The average force $F$, acting for a very short time $\Delta t$, gives the ball an initial horizontal linear momentum $p_0 = F \cdot \Delta t$, and also a (clockwise) initial angular momentum about the CM $L_0 = \tau \cdot \Delta t$, where $\tau = (h - R)F$ is the torque of $F$ about the CM.

a. Let the initial CM speed immediately after the blow by the stick be $v_0$. What is the initial angular speed $\omega_0$ of rotation about the CM at that time, in terms of $h$, $R$ and $v_0$? [Relate $L_0$ to $p_0$.]

b. If the ball is to start out rolling without slipping, what must $h$ be?

c. Suppose $h = R$. Then the ball will not roll initially, but after skidding for a time it will begin to roll. Use conservation of angular momentum about a point on the table to find the CM speed when it does roll.

d. For the ball to move to the left when it finally does roll, the total angular momentum about the point on the table must be negative. Show that there is no value of $h$ for which this happens.

[20 points]

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a. We see that $L_0 / p_0 = (h - R)$, so using $L_0 = I\omega_0 = \frac{2}{5}mR^2\omega_0$ and $p_0 = mv_0$

we find $\omega_0 = \frac{5}{2}(h - R)\frac{v_0}{R^2}$.

b. For rolling $v_0 = R\omega_0$. This leads to $h = \frac{7}{5}R$.

c. In general the angular momentum about a point on the table is $L = mvR + I\omega$. In this case $\omega_0 = 0$ so $L = mRv_0$. When the ball rolls $v = R\omega$ and $L = mvR + \frac{2}{5}mR^2 \cdot (v / R) = \frac{7}{5}mvR$. Thus $v = \frac{5}{7}v_0$.

d. In general $I\omega_0 = (h - R)mv_0$, so the initial angular momentum about the point on the table is $L = mRv_0 + (h - R)mv_0 = hm v_0$. This cannot be negative. [To make the ball eventually roll backwards one must strike it with a force having a downward component.]
5. Because Newton did not know the constant $G$ or the mass of the earth $M_e$, he had to find indirect ways to check his law of gravitation. He derived a relation between the acceleration of gravity at earth’s surface $g$, the period $T$ and radius $a$ of the moon’s orbit, and the radius of the earth $R$. All of these quantities were known with reasonable accuracy in his time. You are to make that derivation, using only the gravitation law, the 2nd law, and the facts about circular motion.

a. First, analyze the force on a small mass $m$ near the surface of the earth to derive a formula giving $g$ in terms of $G$, $M_e$ and $R$.

b. Next, relate $a$ and $T$ to $G$ and $M_e$, assuming the moon’s orbit is a circle. [This is Kepler’s 3rd law, but you are to derive it for this case from the listed principles.]

c. Eliminate $G$ and $M_e$ between the relations you have derived to find the desired formula for $g$ in terms only of $a$, $T$ and $R$:

$$g = 4\pi^2 \frac{a^3}{R^2 T^2}.$$  

[15 points]

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\[\begin{align*}
\text{a.} & \quad \text{From the 2nd law we have} \quad \frac{GM_em}{R^2} = mg, \quad \text{so} \quad g = \frac{GM_e}{R^2}. \\
\text{b.} & \quad \text{The radial acceleration is} \quad a_r = a\omega^2 = a\left(\frac{2\pi}{T}\right)^2 = 4\pi^2 \cdot a / T^2, \quad \text{so for the moon the 2nd law is} \quad \frac{GM_e M_m}{a^2} = M_m \cdot 4\pi^2 \frac{a}{T^2}. \quad \text{This gives} \\
& \quad a^3 / T^2 = GM_e / 4\pi^2. \\
\text{c.} & \quad \text{Combining to eliminate} \quad G \quad \text{and} \quad M_e, \quad \text{we find the result claimed.} \\
& \quad \text{[When Newton first did this calculation the numbers didn’t quite work out because the measured value of} \quad R \quad \text{was a bit wrong. By the time he wrote the} \quad \textit{Principia} \quad \text{better measured values had been obtained and the numbers fit perfectly.]} \\
\end{align*}\]
Median = 50.5
SD = 16.0

Median = 50.8
SD = 14.6