

Physics 53

Oscillations

You've got to be very careful if you don't know where you're going, because you might not get there.

— Yogi Berra

Overview

Many natural phenomena exhibit motion in which particles retrace the same trajectory repeatedly. This motion is called *periodic* motion, or *oscillation*.

The simplest number characteristic of this motion is the time it takes the particle to make one complete trip through the trajectory, returning to the initial point with the initial velocity. This time is the **period**, usually denoted by T . Alternatively, one can use the number of complete trips made per unit time, called the **frequency**, denoted by f . Clearly $f = 1/T$.

If $\mathbf{r}(t)$ is the position of the particle, and $\mathbf{v}(t)$ its velocity, then for oscillation with period T we must have *both* $\mathbf{r}(t+T) = \mathbf{r}(t)$ and $\mathbf{v}(t+T) = \mathbf{v}(t)$. That is, the particle must return, after time T , to its original position *and* velocity.

It is important that *both the position and velocity* return to the original values. In the swing of a pendulum, the bob passes through the bottom point *twice* in each oscillation, once with the original velocity and once with the same speed but opposite direction.

An important type of periodic motion is **harmonic motion** in which the time dependence is sinusoidal:

$$\mathbf{r}(t) = \mathbf{r}_0 \cos \omega t,$$

where \mathbf{r}_0 is a constant (vector) and $\omega = 2\pi f = 2\pi/T$. The constant \mathbf{r}_0 , giving the maximum value of \mathbf{r} , is the **amplitude**; the constant ω is the **angular frequency**.

The choice of cosine rather than sine is arbitrary. Our choice makes $\mathbf{r}(0) = \mathbf{r}_0$.

The special case where ω is independent of the amplitude is called **simple harmonic motion** (SHM). We begin with that case, in one dimension for simplicity.

Simple harmonic motion

We restrict the motion to the x -axis, so the most general SHM is described by:

Simple harmonic motion (1-D)	$x(t) = A \cos(\omega t + \phi)$
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Here A is the amplitude, ω is the angular frequency, and ϕ is called the “initial phase”.

The term “phase” in harmonic motion means the argument of the cosine, in this case the quantity $(\omega t + \phi)$. The constant ϕ is the value of the phase at $t = 0$.

The velocity and acceleration are found as usual by taking time derivatives:

$$v(t) = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$a(t) = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi)$$

We see that $a = -\omega^2 x$. What sort of force gives rise to this motion? From the 2nd Law the total force must have the general form

$$F = -m\omega^2 x.$$

If the total force is a negative constant times x , the resulting motion is SHM and the constant is $m\omega^2$.

One can sometimes use this fact to determine ω from a knowledge of the force.

Forces obeying Hooke's law, such as a mass attached to an ideal spring ($F = -kx$) are of this type and give rise to SHM. We find a useful formula for that case:

$$\omega = \sqrt{\frac{k}{m}}.$$

The potential energy for the force $F = -m\omega^2 x$ is easily obtained:

$$U(x) = \frac{1}{2} m\omega^2 x^2.$$

In any situation where the (total) potential energy is a positive constant times x^2 the motion is SHM and the constant is $\frac{1}{2} m\omega^2$.

This allows determination of ω from a formula for $U(x)$. For the mass-spring system $U(x) = \frac{1}{2} kx^2$.

In SHM kinetic and potential energy both oscillate with time, while their sum remains constant (since the force is conservative). Twice per cycle the energy is entirely kinetic, and twice it is entirely potential. The total energy is the maximum potential energy:

Total energy in SHM	$E = \frac{1}{2}m\omega^2 A^2$
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The initial phase ϕ is determined from the initial position and velocity. There are two common cases:

- The particle starts from rest at $x = A$. Then $\phi = 0$ and $x = A\cos\omega t$.
- The particle starts from $x = 0$ with velocity v_0 . Then $\phi = -\pi/2$, $x = A\sin\omega t$, and $v_0 = \omega A$.

Small oscillations about equilibrium

As we have seen, if a system's (total) potential energy has the parabolic form $U(x) = Cx^2$, where C is a positive constant, then we have SHM, and $C = \frac{1}{2}m\omega^2$.

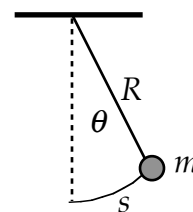
No actual system could have this potential energy for all values of x because its potential energy would become infinite. But in the vicinity of a minimum *any* curve is *approximately* parabolic, so if we restrict x to small values about a *minimum* in U the motion will be *approximately* SHM. Since a minimum in U is also the location of a point of stable equilibrium, we have a general rule:

Small oscillations about a point of stable equilibrium are approximately SHM.

If the total energy E is only slightly greater than the value of U at the minimum, the turning points in the motion will occur close to the location of the minimum, and the SHM approximation will give good results. This is the case of **small oscillations**.

An example is the swing of a pendulum for small amplitudes. The situation is as shown. Choosing $U = 0$ at the bottom of the swing, the potential energy in the configuration shown is $U(\theta) = mgR(1 - \cos\theta)$. We

wish to put this in terms of the distance s traveled along the arc of the circle. Since arc length is related to the angle (in radians) by $s = R\theta$ we can write



$$U(s) = mgR[1 - \cos(s/R)].$$

This is *not* in the form of a positive constant times s^2 , so the motion is not exactly SHM. But if s is much smaller than R we can apply the small angle approximation for the cosine:

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \text{ for } \theta \ll 1 \text{ (radians).}$$

We then find for $s \ll R$

$$U(s) \approx \frac{1}{2} m \frac{g}{R} s^2,$$

which has the required parabolic form for the potential energy in SHM. For small amplitude oscillations, the motion is thus *approximately* SHM, with angular frequency

$$\omega = \sqrt{\frac{g}{R}}.$$

Note that the frequency depends on R and g , but *not* on m .

Through this formula, one can use a pendulum to measure the value of g . By the late 1600's pendulum clocks were good enough that it was possible to measure variations in g from place to place on the earth by the effect they had on the time kept by the clock. This is how the increase of the earth's radius at the equator was discovered.

Because the frequency also depends on the length, changes in R due to thermal expansion or contraction must be compensated if a pendulum clock is to remain accurate at different temperatures.

That for small swings the period of a pendulum does not change as the amplitude decreases was discovered by Galileo.

Damped oscillations

Realistic oscillations in a macroscopic system are subject to dissipative effects, such as friction, air resistance, and generation of heat as a spring stretches and compresses repeatedly. These losses steadily decrease the energy of the oscillating system, reducing the amplitude of the oscillations, a phenomenon called **damping**.

The rate at which energy is lost is described by the *Q-value*, defined as

$$Q = 2\pi \left(\frac{E}{\Delta E} \right).$$

Here $\Delta E / E$ is the fractional energy loss per cycle of the oscillation. The *larger* Q is the *less* important the energy losses are.

The symbol Q stands for "quality", since in some oscillating systems (a bell, for example) it is desirable that the system oscillate (ring) for a long time.

Damping can arise from many causes. We will analyze the mathematically simple case where the dissipative force depends on the speed of the oscillating particle. This case is called “viscous” damping. Air resistance is an example.

For relatively low speeds the damping force is approximately linear in v , so we write

$$F_d = -bv,$$

where b is a positive “damping” constant.

This force acts together with the force responsible for the oscillation, $F_{\text{osc}} = -m\omega_0^2 x$, where ω_0 is what the angular frequency would be if there were no damping. From the 2nd Law we then have

$$ma = -m\omega_0^2 x - bv.$$

Writing a and v in terms of time derivatives of x and rearranging, we have

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \omega_0^2 x = 0.$$

This is a differential equation to be solved for $x(t)$. Depending on the values of the constants, there are three types of solutions, which we give without proof

Underdamped case ($\omega_0 > b / 2m$): The system oscillates, but with steadily decreasing amplitude. The solution is

$$x(t) = Ae^{-(b/2m)t} \cos(\omega t + \phi).$$

Here A and ϕ depend on how the oscillation is started. The angular frequency of the damped oscillation is smaller than ω_0 :

$$\omega = \sqrt{\omega_0^2 - (b / 2m)^2}.$$

Overdamped case ($\omega_0 < b / 2m$): In this case there is no oscillation. The system's original displacement simply dies away to zero according to the formula

$$x(t) = A_1 e^{-\alpha_+ t} + A_2 e^{-\alpha_- t}.$$

The A 's are determined by initial conditions, and the (positive) α 's are given by

$$\alpha_{\pm} = (b / 2m) \pm \sqrt{(b / 2m)^2 - \omega_0^2}.$$

Critically damped case ($\omega_0 = b / 2m$): The displacement dies away without oscillating, but here the formula is

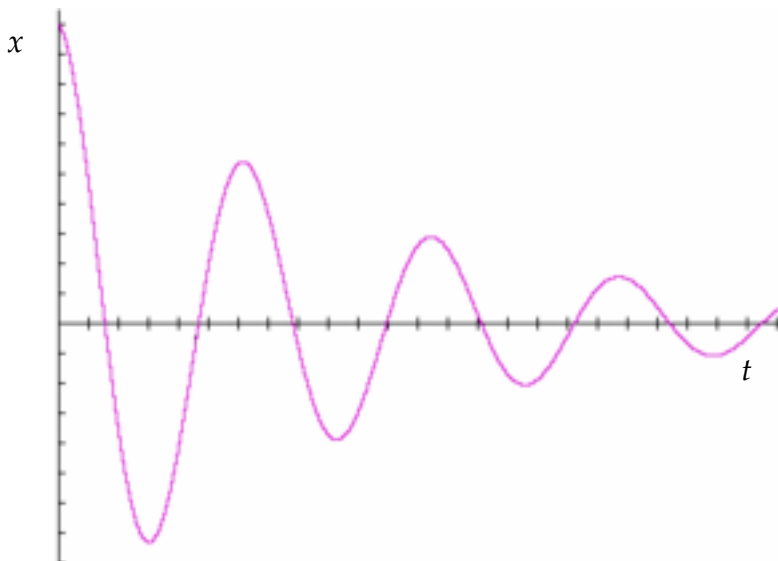
$$x(t) = Ae^{-\omega_0 t} (1 + \omega_0 t).$$

In this case the displacement goes toward zero at the fastest possible rate. .

That these expressions for $x(t)$ solve the differential equation is not hard to show by direct substitution.

The underdamped case is of most interest to us. Shown is x vs. t for a typical case. For small damping (large Q) it is easy to show that the Q -value is given by

$$Q = \frac{m\omega_0}{b}.$$



Driven oscillations

Damping removes energy from the system, so for the oscillation to continue indefinitely some external agent — a **driving force** — must put energy in. If the energy lost to damping in each cycle is exactly compensated by work input from the external “driving force” then we have a **steady state** situation.

We will treat the (mathematically) simplest driving force, described by

$$F_{\text{ext}} = F_0 \sin \omega t.$$

This is added to the forces considered above in the damped oscillator case. The differential equation for $x(t)$ becomes

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m} \sin \omega t.$$

We are interested only in the steady state solution, in which the system oscillates at the frequency of the driving force. This solution must have the standard form for SHM, which we will write for later convenience as

$$x(t) = A_0 \sin(\omega t - \phi_0),$$

where A_0 and ϕ_0 are to be determined by the requirement that this function satisfy the differential equation. Direct substitution and much algebra lead to

$$A_0 = \frac{F_0 / m}{\sqrt{(\omega^2 - \omega_0^2)^2 + (b\omega / m)^2}}$$

$$\tan \phi_0 = \frac{(b\omega / m)}{\omega^2 - \omega_0^2}.$$

The power input by the driving force is $P_{\text{in}} = F_{\text{ext}}v$, where $v = \omega A_0 \cos(\omega t - \phi_0)$. This might be a complicated function of time, but its average over a cycle is relatively simple:

$$P_{\text{av}} = \frac{F_0^2}{2b} \sin^2 \phi_0.$$

This formula shows two important properties of the driven oscillator:

- The power input is inversely proportional to the strength of the damping.
- For variable driving frequency ω the power input has its maximum value when $\phi_0 = \pm\pi/2$, or $\tan \phi_0 \rightarrow \pm\infty$. From the formula above, we see that this occurs when $\omega = \omega_0$. This situation is called **resonance**. It is an important property of driven oscillations.

Shown below are two curves of P_{av} vs. ω , for the same values of F_0 and ω_0 , but for two values of b differing by a factor of 4. Small damping gives a sharp and high resonance peak, while large damping gives a broad and moderate resonance peak.

