Phases of matter

Until now we have analyzed mostly systems in which the particles have a fixed average spatial relation to each other, i.e., are bound. This kind of system is a solid.

If the particles are not bound, but still are on average close enough together to interact continuously with nearest neighbors, we have a liquid.

When the particles are on average far apart and only interact occasionally when they collide with each other, we have a gas.

These situations define the three commonly occurring phases of matter.

In the liquid and gas phases, the particles move about (flow) until the system takes the shape of the solid container surrounding them. Such systems are called fluids. They are the object of our study here. For the most part we will discuss liquids first, and later discuss gases in the context of thermal physics.

Macroscopic variables

It would be hopeless to try to describe a fluid by listing the detailed behavior of every particle, because there are so many. We must seek a simpler description, in which the main focus is on average behavior of the particles. The variables describing the average (or bulk) behavior are called macroscopic variables. They are distinguished from microscopic variables, which describe the detailed behavior of individual particles. Ordinary measuring devices tell us only the values of the macroscopic variables.

For fluids, the most useful macroscopic variables are the volume occupied by the fluid, the pressure in the fluid, and the (average) velocity of the particles in a small region of the fluid (which nevertheless contains a very large number of particles). Since pressure and velocity generally vary from place to place in the fluid, they are fields; pressure is a scalar field and velocity is a vector field.
**Stresses in a fluid**

Earlier we discussed elastic properties of solids in terms of stresses and strains. There are two important differences when fluids are considered:

- Fluids do not resist shear stresses, but simply flow if forces parallel to the surface are applied. Indeed, a fluid is often defined as a system that cannot offer resistance to shear stresses.

- Compressive or tensile stresses may result in a change in volume of the fluid, although for many liquids (such as water) the change in volume may be quite small. A tensile stress at the fluid surface, tending to pull the particles at the surface back into the fluid, is called **surface tension**. The compressive stress within the fluid, pushing the particles outward toward the surface, is called the **pressure**. The shape of a volume of liquid not constrained by walls of a container — a water drop, for example — is determined by equilibrium between these two stresses.

**Pressure**

Consider a fluid at rest. By “at rest” we mean that the velocity field in the fluid is zero at all points. This does not mean that the microscopic particles themselves are at rest; indeed, they are constantly and rapidly moving. But this motion of the particles is random as to direction, so that statistically their velocities average to zero.

A “point” in the fluid, from the *macroscopic* point of view, is actually a small region that contains a very large number of *microscopic* particles, and the velocity at that “point” is the average over those particles.

Let there be a solid object submerged in this fluid. The microscopic fluid particles near the submerged object collide with its surface a large number of times per second. Each collision transfers a small amount of momentum from the particle to the submerged object, and the cumulative effect of this momentum transfer gives rise to an average force on the object. It is the distribution of that average force over the surface of the object that we describe by the pressure field on that surface.

The operational definition of pressure is this:

<table>
<thead>
<tr>
<th>Pressure in a fluid</th>
<th>At a point on a body immersed in a fluid, let the force exerted by the fluid normal to a small area $A$ of the surface of the body be $F_\perp$. Then the pressure in the fluid at that point is given by $P = F_\perp / A$.</th>
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Analysis of a simple model — the particles of the fluid are taken to be mass points and the collisions with the surface are assumed to be elastic — gives the following linear relation between pressure and average kinetic energy $K_{av}$ of the particles:

$$P = \frac{2}{3} n K_{av},$$

where $n$ is the number of fluid particles per unit volume.

It is important to remember that pressure described here arises from the effects of the random motion of the fluid particles. One sometimes speaks of the “pressure” exerted on something by a rapidly moving fluid (a strong wind, or water from a fire hose, for example). There can obviously be a force per unit area on an object in the path of the fluid in such cases. But the momentum transfer giving rise to that force arises from the non-zero average velocity of the fluid particles. What we are treating here is the effect of random motion of the fluid particles, for which the average velocity is zero. It is often called the “hydrostatic pressure,” especially when dealing with water.

### Effects of gravity

If the fluid is in the earth’s gravitational field then the average kinetic energy of the particles will vary with height because of conservation of energy. Deeper in the fluid, where there is less gravitational potential energy, there will be greater average kinetic energy, hence greater pressure. This variation with depth is fairly easy to analyze.

Consider the small cube of the fluid shown. Let the mass density (mass per unit volume) of the fluid be $\rho$. Then the mass of the cube of fluid is $dm = \rho dx dy dz$. At any point $(x, y, z)$ let the pressure field be $P(x, y, z)$. We are interested in the net forces on the cube faces:

- In the $x$-direction: $F_x = [P(x) - P(x + dx)] dy dz$.
- In the $z$-direction: $F_z = [P(z) - P(z + dz)] dx dy$.
- In the $y$-direction: $F_y = [P(y) - P(y + dy)] dx dz - g dm$.

If the fluid in the cube is in static equilibrium, all these force components must be zero. This means that the pressure does not change in the $x$ or $z$ directions, while in the $y$ direction we have

$$P(y) - P(y + dy) = g \frac{dm}{dx dz} = \rho g dy.$$

Dividing by $dy$ and using the definition of a derivative, we find

$$\frac{dP}{dy} = -\rho g.$$
To find $P$ from this formula, we need to know the dependence of $\rho$ on either $y$ or $P$. We will examine two common cases.

The simplest case is where $\rho$ is a constant, independent of both $P$ and $y$. A fluid of this type is **incompressible**. (In effect, it has infinite bulk modulus.) In this case we can easily integrate the above equation to find an important formula:

\[
P(y) = P_0 - \rho g y
\]

Here $P_0$ is the pressure at the height where $y = 0$. Many liquids, including water, are nearly incompressible, so this is a useful case.

Suppose we choose $y = 0$ at the top surface of the fluid. If an external force is applied at that surface, changing the value of $P_0$, we see that the pressure at all other levels within the fluid changes *by the same amount*. That an externally applied pressure is “transmitted” equally to all parts of the fluid is **Pascal’s Law**.

Now we consider is a gas, for which the density is approximately proportional to the pressure: $\rho = KP$, where $K$ is a constant. Integrating Eq (1) above we find

\[
P(y) = P_0 e^{-Kg y}.
\]

This is called the “law of atmospheres”, since it gives a useful approximation for the decrease in pressure with height in the earth’s atmosphere.

**Buoyancy**

Consider a certain volume $V$ of a fluid at rest. Because the fluid is at rest, the weight of the fluid within $V$ must be supported by an upward force. This force arises from variations in pressure at the surface of $V$, due to the fluid outside $V$. Because of gravity, the pressure on the lower surfaces (which gives rise to upward forces) is greater than the pressure on the upper surfaces (which gives rise to downward forces). The net upward force arising from these pressure differences is the **buoyant force**. It must have magnitude exactly equal to the weight of the fluid in $V$ in order that $V$ remain at rest.

If the volume in question is instead occupied by some solid object, the distribution of pressures at the surface from the rest of the fluid remains the same, and results in the same buoyant force, tending to push the object upwards.

This is a discovery famous in the history of ancient science:
**Principle of Archimedes**

An object immersed in a fluid experiences a buoyant force in the direction opposite to gravity, of magnitude equal to the weight of fluid displaced by the body.

The source of this force is the increase in pressure with depth in the fluid, which we have seen is an effect of gravity: the buoyant force is indirectly caused by gravity.

In non-inertial frames, the buoyancy effect is that of effective gravity: the buoyant force is the effective weight of the fluid displaced, and its direction is opposite to $g_{eff}$.

If the object is less dense than the fluid, the buoyant force will be greater than its weight; the object will rise to the surface and float with just enough of its volume below the liquid surface to produce a buoyant force equal to its weight. If the object is more dense than the fluid, it will sink to the bottom, but because of the buoyant force the “apparent weight” of the object — the force exerted on it by the bottom of the container when it is resting there — will be less than its actual weight.

**Fluids in motion**

To describe fluids at rest, the only macroscopic variables we used were pressure, density and volume. If the fluid also undergoes bulk motion, i.e., the average velocity of the particles is not zero, then we must specify that velocity at each point in the fluid. This macroscopic variable is called the velocity field. In general the velocity field is quite complicated. To simplify things we restrict our attention to liquids flowing through pipes or other fixed structures. We further assume that the lines of the velocity field (curves drawn tangent to the direction of the velocity at each point) are smooth, with adjacent layers (“lamina”) of the fluid flowing smoothly beside each other. This motion is called laminar flow.

It is often called “streamline” flow, because the smooth lines of the velocity field are called streamlines.

An important principle here is conservation of mass. The total mass of fluid coming into a given region per unit time must equal the mass leaving it, since mass is neither appearing nor disappearing in the region.

The rate of flow of mass past a given point (mass per unit time) is the mass flux:

\[
\Phi = \frac{dm}{dt}.
\]

In the figure, the amount of mass $\Delta m$ flowing across the area $A$ in time $\Delta t$ is equal to the amount contained in the shaded cylinder, $\Delta m = \rho A v \Delta t$. The mass flux is thus
\[ \Phi = \frac{\Delta m}{\Delta t} = \rho A v. \]

Since mass is conserved, the flux is constant, which gives us an important formula:

| Equation of continuity | \( \rho A v = \text{const.} \) |

In most of the cases we will consider, the fluid is approximately incompressible, so \( \rho \) is treated as constant, and the continuity equation becomes simply

\[ A v = \text{const.} \]

This shows that if an incompressible fluid is forced to pass through a smaller area, the speed correspondingly increases. A common example of this is a nozzle.

**Viscosity and turbulence**

In a real fluid there is some dissipation of energy caused by non-conservative interactions between the pipe walls and the fluid flowing next to the walls, and by non-conservative interactions between the lamina. The lamina at the wall slow until they are nearly at rest. The lamina next to those are partly slowed by interaction with the previous ones, and so on. This results in a “profile” of speeds for the lamina, with those near the walls having the slowest speed and those farther from the walls moving faster.

This dissipative phenomenon is called viscosity. The resulting loss of kinetic energy causes a drop of pressure in the direction of the flow. For the case of flow through a pipe of circular cross section, this pressure drop is given by Poiseuille's law:

\[ \Delta P = \frac{8 \eta L \Phi}{\pi r^4 \rho}. \]

Poiseuille, a French physician, discovered this law experimentally around 1840.

Here \( L \) is the length of pipe, \( r \) is the radius of the pipe, \( \eta \) is the “viscosity” of the fluid, \( \Phi \) is the mass flux, and \( \rho \) is the mass density. This formula allows determination of the external pressure necessary to push fluid through a given length of pipe at a given rate. The most striking feature is the dependence on \( r^{-4} \).

One common application of this law is to blood flow in arteries. We see that if the artery radius is reduced by a factor of 2 the blood pressure difference necessary to maintain the flow rises by a factor of 16.

If the fluid moves too quickly, the lamina break up and form “eddies”. The result is turbulent flow. Much of the forward kinetic energy of the fluid is diverted into rotational energy in the eddies. The resulting “drag” force on an object around which the fluid moves increases dramatically.
**Ideal fluids and Bernoulli’s theorem**

To analyze viscous and (especially) turbulent fluid flow is very difficult mathematically, so we will make some further approximations:

- Energy losses through viscosity are negligible.
- The flow is laminar (no turbulence).
- The fluid is incompressible.

These approximations define an **ideal fluid**.

We will calculate the total work done by the various forces that act on a bit of the fluid as it flows in the pipe segment shown, and apply the work-energy theorem.

There are only two forces doing work:

- Gravity, if the height of the pipe changes.
- The forces due to pressure from parts of the fluid external to the segment under consideration.

Consider the power input by the forces on a bit of mass $m$ as it moves from point 1 to point 2. The pressure at point 1 is $P_1$ and that at point 2 is $P_2$. At those points the average speed of the fluid is $v_1$ and $v_2$ respectively. At point 1 the force $P_1 A_1$ due to the external pressure is parallel to the velocity (both are to the right in the diagram), so the power input ($P = F \cdot v$) from the external pressure is $P_1 A_1 v_1$. At point 2 the force is opposite to the velocity, so the power input is $-P_2 A_2 v_2$.

Using the fact that $A v = \Phi / \rho$ is the same everywhere (continuity), we have for the total power input from pressure forces:

$$\text{Power from external pressure} = (P_1 - P_2) \frac{\Phi}{\rho}.$$  

The power input by gravity is the rate at which gravity does work, which is minus the rate of change of the gravitational potential energy, so

$$\text{Power from gravity} = -\frac{d}{dt} (mg y_2 - mg y_{12}) = \Phi g (y_1 - y_2).$$
(We have used the fact that $\Phi = dm / dt$.)

By the work-energy theorem, the total power input by all forces is the rate of change of kinetic energy, so we have

$$\frac{\Phi}{\rho}(P_1 - P_2) + \Phi g(y_1 - y_2) = \frac{d}{dt}\left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right) = \frac{1}{2} \Phi \left(v_2^2 - v_1^2\right).$$

Canceling common factors and rearranging, we have an important result:

| Bernoulli’s theorem | $P_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2$ |

This has the form of a conservation law, saying that the sum of these three terms is the same everywhere in the pipe. It is very useful in practice — but it must be remembered that it applies only to an *ideal* fluid, so it is always an approximation.

Some aspects of Bernoulli’s theorem:

- For fluids at rest ($v = 0$ everywhere) it gives the variation of pressure with height derived earlier.
- For motion at the same height, it shows that greater fluid speed means lower pressure and vice versa. [Higher fluid speed means more kinetic energy of “bulk” motion, and hence less kinetic energy of random motion, therefore less pressure.]
- Energy can be transferred among pressure, kinetic energy of flow, and gravitational potential energy. One can say that pressure $P$ represents kinetic energy per unit volume of the random motion of the particles, $\frac{1}{2} \rho v^2$ represents kinetic energy per unit volume of the bulk motion, and $\rho g y$ represents potential energy per unit volume. Then Bernoulli’s theorem is simply a statement of conservation of energy.

Bernoulli’s theorem is sometimes invoked in cases where the fluid is a gas, to explain such things as the curving trajectory of a spinning ball as it flies through the air. An argument of that kind should be taken as only qualitatively useful, since gases are certainly not incompressible, and effects of viscosity and turbulence are often very important. Applications of Bernoulli’s theorem to the flow of water through pipes are more quantitatively reliable.