Supplementary Problems for Topics II — Solutions

1. A proton of mass $m$ at rest is struck by a deuteron of mass $2m$ moving along the $+x$-axis with speed $v_0$. The collision is elastic, and the final speed of the proton is twice the final speed of the deuteron.

a. Find the final speed of the proton in terms of $v_0$.

$$v = 2\sqrt{3}v_0$$

b. After the collision the proton moves at angle $\theta$ above the $x$-axis and the deuteron at angle $\phi$ below that axis. Find these two angles. [The answers are numerical.]

$$\cos \theta = v_0 /v = 3 /2$$

$$\phi = \theta = 30^\circ$$

2. A putty ball of mass $m$ moving horizontally as shown with speed $v_0$ strikes the end of a hanging rod of length $R$, pivoted at its top. The rod has mass $3m$ and its moment of inertia about the pivot is $I = mR^2$. The pivot is frictionless. The collision is inelastic, with the putty sticking to the rod.

a. Find the final angular speed $\omega$ of the system’s rotation immediately after the collision. [What is conserved?]

$$\omega = v_0 /2R$$

b. What fraction of the initial kinetic energy of the putty is lost in the collision?

$$\text{Final } K = \frac{1}{2}(I + mR^2)\omega^2 = \frac{1}{4}mv_0^2.$$ Half of the original kinetic energy is lost in the collision.

[The final linear momentum is that of the rod’s CM plus that of the putty: $p_f = (3m)(R /2)\omega + mR\omega = \frac{5}{2}mR\omega = \frac{5}{4}mv_0$. This is larger than the initial momentum, showing that the force exerted by the pivot during the collision is to the right.]
3. The system shown (like a yo-yo) consists of two identical larger wheels connected by a small cylindrical shaft. A string is wrapped around the shaft, and the system can be made to move on the rough floor by pulling on the string. The string makes angle $\theta$ with the horizontal. The system has mass $m$ and moment of inertia $I$ about the symmetry axis.

a. If $\theta = 0$, find the magnitude and direction of the acceleration of the CM. [Find the friction force.]

b. Repeat for $\theta = 90^\circ$.

c. There is a value of $\theta$ for which the acceleration is zero. Consider torques about the point of contact with the floor to find that angle.

\begin{align*}
\text{a. } & \text{ Static friction force is to the left, so we have } T - f_s = ma. \text{ Torques about CM: } R f_s - r T = I \alpha. \text{ Rolling: } \alpha = a / R. \text{ Solve for } a: a = T \frac{1 - r / R}{m + I / R^2}.

\text{b. } & \text{ Again static friction is to the left (opposing slipping of the rotating yo-yo), so we have } f_s = ma \text{ and } r T - R f_s = I \alpha. \text{ Solve for } a: a = T \frac{r / R}{m + I / R^2}.

\text{c. } & \text{ If the line of the string pass through the point of contact, then the tension give no torque about that point. Neither gravity nor friction give torques about that point, so the total torque would be zero and the yo-yo would not rotate. As the drawing shows, the angle in that case is given by } \\
& \cos \theta_0 = r / R.
\end{align*}
4. A system consisting of two balls of equal mass \( m \) attached to the ends of a massless rigid rod of length \( L \) is in free fall through empty space as shown toward a spherical star of very large mass \( M \). The star's center is at distance \( r \) from the center of the rod.

a. Let the gravitational force exerted by \( M \) on the left ball be \( F_1 \) and that on the right ball \( F_2 \). If the system holds together so that the two balls have the same acceleration, what tension \( T \) must the rod provide, in terms of \( F_1 \) and \( F_2 \)? [Draw free-body diagrams for each of the two balls.]

b. Express the two forces in terms \( G, M, m, L \) and \( r \) to find \( T \). Then find the approximate formula for \( T \) in the case where \( L \ll r \).

[Use the binomial approximation that \( 1/(1+x)^2 \approx 1-2x \) for small \( x \). In this approximation, \( T \) should be proportional to \( 1/r^3 \).]

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<tr>
<th>( F_1 )</th>
<th>( F_2 )</th>
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<tr>
<td>( \frac{GMm}{r^2(1-L/2r)^2} )</td>
<td>( \frac{GMm}{r^2(1+L/2r)^2} )</td>
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On the left ball we have \( F_1 - T = ma \), and on the right ball \( F_2 + T = ma \), so

\[
T = \frac{1}{2}(F_1 - F_2).
\]

Here \( F_1 = GMm \frac{1}{(r-L/2)^2} \) and \( F_2 = GMm \frac{1}{(r+L/2)^2} \). We write

\[
F_1 = \frac{GMm}{r^2} \frac{1}{(1-L/2r)^2} = \frac{GMm}{r^2} (1-L/2r)^{-2} = \frac{GMm}{r^2} (1+L/r). \text{ Similarly,}
\]

\[
F_2 = \frac{GMm}{r^2} \frac{1}{(1+L/2r)^2} = \frac{GMm}{r^2} (1+L/2r)^{-2} = \frac{GMm}{r^2} (1-L/r). \text{ Thus}
\]

\[
T \approx \frac{GMmL}{r^3}.
\]

[This is an example of tidal forces. From the point of view of the two mass system, there is a gravitational effect pushing them apart that is counteracted by the tension in the rod. This effect is what raises the level of the oceans on opposite sides of the earth due to the moon’s and sun’s gravitational field.]
5. A spaceship of mass \( m = 10^3 \text{ kg} \) is in a circular orbit around the earth as shown. Its orbital speed is 4 km/s.

a. What is the total energy of the orbit? [Express \( E \) in terms of the kinetic energy.]

The engines of the spaceship are now fired, sending it into an elliptical orbit with distance of farthest recession from earth equal to \( 3R \).

b. Sketch the new orbit on the drawing.

c. How much energy did the engines put in? [What is the total energy of the new orbit?]

[Except for the drawing, all answers are numerical.]

\[
\begin{align*}
a. \quad & \text{The total energy is } E = -\frac{GMm}{2R}, \text{ while the potential energy is } U = -\frac{GMm}{R}, \\
& \text{so the kinetic energy is } K = E - U = +\frac{GMm}{2R} = -E. \text{ Since} \\
& K = \frac{1}{2}mv^2 = \frac{1}{2}(10^3)(4 \times 10^3)^2 = 8 \times 10^9 \text{ J, we have } E = -8 \times 10^9 \text{ J.} \\
b. \quad & \text{Drawing shown. The ellipse must pass through the present location of the spaceship, and must go outside the original orbit.} \\
c. \quad & \text{Now the total energy is } E' = -\frac{GMm}{4R} = -4 \times 10^9 \text{ J. The energy input from the engines is } E' - E = 4 \times 10^9 \text{ J.}
\end{align*}
\]