Assignment 8

1. Some questions about the vector product \( \mathbf{C} = \mathbf{A} \times \mathbf{B} \).
   a. Prove the rules given in the notes for the magnitude and direction of \( \mathbf{C} \) from the definition in terms of components. Let \( \mathbf{A} \) and \( \mathbf{B} \) lie in the \( x-y \) plane with \( \mathbf{A} \) along the \( x \)-axis and \( \mathbf{B} \) having direction with angle \( \theta \) relative to \( \mathbf{A} \).
   b. One requires \( \theta \) to be the angle between \( \mathbf{A} \) and \( \mathbf{B} \) that is no larger than \( \pi \). Why?
   c. What are the rules for the vector products of the unit vectors \((\mathbf{i}, \mathbf{j}, \mathbf{k})\)?
   d. Prove directly from the definition in terms of components that \( \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \). [Use the situation in (a).]

2. Consider the effects of the gravitational force on the particles of a system. The \( i \)th particle, of mass \( m_i \) at position \( r_i \) experiences a gravitational force \( m_i \mathbf{g}(r_i) \), where \( \mathbf{g}(r_i) \) is the gravitational field at that point. If the gravitational field is uniform, then \( \mathbf{g}(r_i) \) is the same at the location of all the particles. Call it \( \mathbf{g} \).
   a. If the gravitational field is uniform, show that the total gravitational force on the system is simply \( M \mathbf{g} \), where \( M \) is the total mass.
   b. The gravitational torque about the CM from the force on the \( i \)th particle is \( \mathbf{\tau}_i = r'_i \times m_i \mathbf{g}(r_i) \), where \( r'_i \) is the position of the particle relative to the CM. Show that if the gravitational field is uniform the total gravitational torque about the CM is zero.
3. Comment on the validity of these statements about forces and torques.
   a. If two forces obey $F_1 + F_2 = 0$, then they give zero torque about any point.
   b. If the total force on a body is zero and the total torque from those forces is zero about some point, then the total torque from those forces is zero about all points.
   c. Two forces acting along two separate parallel lines obey $F_1 + F_2 = 0$. There is a point between those lines about which the total torque is zero.
   d. If $F_1 + F_2 = 0$ and the forces both act along the same line, then they give zero total torque about any point.

4. An object with a circular cross section of radius $R$ and moment of inertia about its symmetry axis $I = \beta m R^2$ is rolling down an incline of angle $\theta$. The coefficient of static friction is $\mu_s$.
   We wish to find the maximum value of $\theta$ for which the object will not slip as it rolls.
   a. Assume the object does not slip, so that static friction acts at the point of contact. If the object starts from rest as shown, what is the direction of the static friction force? [What direction (clockwise or counter-clockwise) must the angular acceleration $\alpha$ be?]
   b. Write the 2nd law for the acceleration $a$ down the incline, in terms of the static friction force $f_s$.
   c. Write the rotational 2nd law in terms of $I$, $\alpha$, and $f_s$.
   d. Impose the rolling condition relating $a$ to $\alpha$.
   e. Eliminate $a$ and solve for $f_s$ in terms of $\theta$ and known constants.
   f. Find the normal force $N$.
   g. Use $f_s \leq \mu_s N$ to find the maximum value of $\theta$ in terms of $\mu_s$ and $\beta$. \text{Ans:} $\tan \theta \leq \mu_s \cdot \frac{\beta + 1}{\beta}$.
   h. For a hoop $\beta = 1$, for a solid cylinder $\beta = 1/2$, and for a solid sphere $\beta = 2/5$. Rank the maximum values of $\theta$ for these objects.
5. In the situation of the previous problem, let the object start from height \( h \) above the bottom of the incline.
   a. If it rolls without slipping from rest, what is its speed at the bottom? \( \text{Ans:} \quad v^2 = \frac{2gh}{1 + \beta} \).
   b. Let \( h = 50 \text{ cm} \) and let the length of the incline be 2 m. You release from rest at the same time a hoop of mass 0.1 kg and radius 2 cm, and a cylinder of mass 0.2 kg and radius 1 cm. Find the time each takes to reach the bottom. [Find the speed at the bottom and the average speed.] \( \text{Ans:} \) Hoop: \( \frac{4}{\sqrt{5}} \approx 1.79 \text{ s} \); cylinder: \( 2\sqrt{3/5} \approx 1.55 \text{ s} \).
   c. Do the same calculation for two cylinders, the one in (b) and one of mass 0.4 kg and radius 3 cm.

6. The yo-yo shown in two views is resting on a floor that has a large enough value of \( \mu_s \) for the yo-yo to roll without slipping. The string is pulled at the angle shown with tension \( T \). The yo-yo has mass \( m \) and moment of inertia \( I \) about its symmetry axis. The radius of the axle is \( r \) and that of the outer rim is \( R \).
   a. If \( \theta = 0 \) what is the acceleration (magnitude and direction) of the CM? [What is the direction of the friction force?] \( \text{Ans:} \quad a = T \cdot \frac{1 - r/R}{m + I/R^2}, \text{ to right.} \)
   b. Repeat if \( \theta = \pi/2 \). \( \text{Ans:} \quad a = T \cdot \frac{r/R}{m + I/R^2}, \text{ to left.} \)
   c. For what value of \( \theta \) is the acceleration zero? \( \text{Ans:} \co \theta = \frac{r}{R}. \)
7. Some questions about rolling.
   a. A ball of mass \( m \), radius \( R \), and moment of inertia \( \frac{2}{5} mR^2 \), rolls without slipping at speed \( v \) on a horizontal floor. It starts up a frictionless incline. How high does it go before starting back down? Ans: \( h = \frac{v^2}{2g} \).

   b. The same ball rolls without slipping on the track shown, moving vertically at the end, at distance \( h \) below the level at which it started from rest. How high does it go in the air before falling? Ans: \( h' = \frac{5}{7} h \).

   c. Now the ball is on the inside of a vertical circular track, rolling up the side as shown. There is enough friction to make it roll without slipping. What direction is the friction force at this point? [Consider the angular acceleration.] Ans: Upward.

8. The ball is now released from rest on the track shown. It rolls without slipping around the inside of the circular portion which has radius \( d >> R \). Find the minimum height \( h \) that will allow the ball to roll across the top of the circle. Ans: \( h = 2.7d \).

9. The door shown has two hinges supporting it. Over time, what happens?
   a. The screws holding the bottom hinge to the door and the wall tend to work loose, because the door pulls on the hinge.
   b. The screws holding the top hinge to the door and the wall tend to work loose, because the door pulls on the hinge.
   c. Both sets of screws tend to pull loose, because the door pulls on both hinges.
   d. There is no force pulling either set of screws loose.
   [Consider torques about the bottom hinge.]