Assignment 13

1. An empty steel drum of mass \( M \), height \( h \) and cross-section area \( A \) is floating as shown in water. The bottom of the drum is at depth \( d \). Give answers in terms of the properties of the drum, plus \( g \) and the density \( \rho \) of water.
   
   a. What is \( d \)?
   
   b. An external force \( F \) pushes the drum vertically down and extra distance \( x_0 \) and hold it there. What is \( F \) in terms of \( x_0 \) and the other quantities?
   
   c. Now the force is suddenly removed. Show that the subsequent vertical motion of the drum is SHM, and find the angular frequency.

2. An ideal massless spring of stiffness \( k \) is attached to a floor and a horizontal plate of mass \( M \). Resting on the plate is a block of mass \( m \).

   a. When the system is at rest, how much is the spring compressed?

   b. The system is pushed down an extra distance \( A \) and released. What is the angular frequency \( \omega \) of the vertical oscillation?

   c. What is the maximum value of \( A \) for which the block will not leave the plate at any point?
3. Questions about oscillations that are not quite simple harmonic.
   a. We have been assuming the springs in our examples have no mass. If we took into account the mass of an actual spring attached to a mass \( m \) and oscillating, would the real value of \( \omega \) be larger or smaller than the one we get from \( \omega = \sqrt{k/m} \)? Explain.
   b. The oscillations of a simple pendulum are SHM in the approximation that \( \cos \theta \approx 1 - \theta^2 / 2 \), where \( \theta \) is the angle made by the string with the vertical. The approximation for the cosine is more exact if we keep the next term in the series, so \( \cos \theta \approx 1 - \theta^2 / 2 + \theta^4 / 12 \). Would this make \( \omega \) larger or smaller than \( \sqrt{g/\ell} \)?

4. A block of mass \( m \) is attached as shown to a horizontal spring, which is attached to a wall. The block is oscillating on the frictionless floor with amplitude \( A_0 \) and maximum speed \( v_0 \).

   At the instant when the block is at its furthest to the right, momentarily at rest, it is struck by a second identical block, moving to the left at speed \( v_0 \), and the two block stick together after the brief collision.
   a. What is the ratio \( \omega / \omega_0 \) of the angular frequencies of oscillation after and before the collision?
   b. What is the ratio \( A / A_0 \) of the amplitudes after and before the collision?
   c. What is the ratio \( v / v_0 \) of the maximum speeds after and before the collision?
6. A spring of stiffness $k$ is attached to a wall and to the axle of a wheel of mass $m$, radius $R$, and moment of inertia $I = \beta m R^2$ about the axle. The spring is stretched distance $A$ and the wheel is released from rest. The floor has sufficient static friction that the wheel rolls without slipping.

   a. When the spring is stretched distance $x$ and the wheel’s CM has speed $v$, what is the total energy of the system? \textit{Ans:} $\frac{1}{2} m (1 + \beta) v^2 + \frac{1}{2} k x^2$.

   b. What is the maximum speed of the CM? \textit{Ans:} $v_{\text{max}}^2 = \frac{k A^2}{m (1 + \beta)}$.

   c. Show that the motion is SHM and find the angular frequency $\omega$. [One way: use $dE / dt = 0$, where $E$ is the total energy, and show that $a = -(\text{const}) \cdot x$, where the constant must be $\omega^2$.] \textit{Ans:} $\omega^2 = k / (m (1 + \beta))$.

7. A small mass sliding without friction on a circular track executes the same motion as the bob of a simple pendulum, the normal force from the track replacing the tension in the string. So it is only approximately SHM for small oscillations near the bottom of the track. But it is possible to design a track in which the mass will execute SHM for large amplitudes. The shape that works is part of a cycloid, a curve given in parametric form by

   $$x = R(t + \sin t), \quad y = R(1 - \cos t).$$

   Here $t$ can be thought of as like the time, and these equations give the trajectory. You are to show that this track gives SHM, and to find the angular frequency.

   a. We need to convert from $x$ and $y$ to the arc length variable $s$ that gives the distance along the track from the bottom. The formula from calculus is

   $$\left( \frac{ds}{dt} \right)^2 = \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2.$$

   Carry out the derivatives to find $ds / dt$ as a function of $t$.

   b. Integrate to find $s(t)$, choosing $s = 0$ when $t = 0$.

   c. Use the identity $\sin^2(\theta / 2) = \frac{1}{2}(1 - \cos \theta)$ to relate $s^2$ to $y$.

   d. The potential energy is $mgy$, of course. Write this in terms of $s$ to show the motion is exactly SHM and read off the value of $\omega$. 

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8. Two blocks, of mass $m$ and $2m$, are resting on a frictionless table, attached as shown to an ideal spring of stiffness $k$. A third block, of mass $m$, moving to the right with speed $v_0$ as shown, collides with and sticks to the other block of mass $m$. The combined system then moves to the right.

[Give all answers in terms of $m$, $k$, and $v_0$.]

a. Describe the subsequent motion of the system.

b. What is the total kinetic energy just after the collision, when the block of mass $2m$ has not yet started to move? [What is conserved in the collision?]
   Ans: $\frac{1}{4}mv_0^2$.

c. What is the kinetic energy at an instant when all the blocks are traveling with the same speed? [This is the kinetic energy of the CM motion alone.]
   Ans: $\frac{1}{8}mv_0^2$.

d. The instant in (c) is when the compression or extension of the spring is its maximum amount $x_{max}$. What is $x_{max}$? Ans: $\frac{v_0}{2}\cdot\sqrt{m/k}$.

9. Two mass-spring systems, $A$ and $B$, are attached to a flexible horizontal rod as shown. When $A$ is set into oscillation with its natural frequency $\omega_0$, the rod begins to vibrate slightly up and down at that frequency. This vibration acts as a driving force for $B$. We are interested in the average power of the driven oscillation of $B$, in two cases: (1) $B$’s natural frequency is $\omega_0$; (2) $B$’s natural frequency is $2\omega_0$. Assume $A$ is undamped.

a. Let $B$ have damping such that $b/m = \omega_0$.
   What is the ratio of the average power delivered to $B$ in the two cases? Ans: $17/4$.

b. Repeat for the case where $b/m = \omega_0/10$. Ans: 226.

c. In case (1) let both systems be undamped. Discuss the energy transfer between the systems over time. (Total energy is conserved.)

[See the notes on Oscillations, page 7.]