Assignment 11

1. Three systems in static equilibrium.
   a. A mobile hangs as shown.
      The strings and crossbars are massless. The strings supporting the crossbars are attached at distance \( \ell / 4 \) from the left end.
      The mass of ball #1 is \( m \). What are the masses of the other balls?
   b. A large ball of mass \( M \) is attached to a string attached to a frictionless vertical wall as shown. The ball has radius \( R \) and the distance \( d \) shown is between the point where the string is attached to the wall and where the ball touches the wall. Find the tension \( T \) in the string and the normal force \( N \) exerted on the ball by the wall. \( \text{Ans:} \)
      \[
      T = mg \cdot \sqrt{1 + \left(\frac{R}{d}\right)^2}, \quad N = mg \cdot \left(\frac{R}{d}\right).
      \]
   c. Shown is a diagram of the foot of a person supporting her weight on the ball of one foot, with the heel slightly off the floor. The ball is \( a \). The leg bone \( b \) is attached to the foot at horizontal distance 15 cm from \( a \), and the calf muscle \( c \) acts at horizontal distance 20 cm from \( a \). The person’s weight is 500 N. What vertical forces (magnitude and direction) act at \( b \) and \( c \)? \( \text{Ans:} \) At \( b \), 2000 N down; at \( c \), 1500 N up.

2. How and how not to get a wheel over a curb. The wheel of a baby’s stroller needs to be lifted over a curb, as shown. Force \( F \) is applied at the axle of the wheel, at angle \( \beta \) above the horizontal. The wheel has mass \( M \) and radius \( R \).
   a. If \( \beta = 0 \) what must \( F \) be to lift the wheel?
      \( \text{Ans:} \) \( mg \cot \alpha \).
   b. What is the value of \( \beta \) for which \( F \) is smallest? \( \text{Ans:} \) \( \beta = \frac{\pi}{2} - \alpha \).
3. Suppose a planet is a uniformly dense sphere of total mass $M$ and radius $R$.
   a. Find the gravitational field $g$ inside the sphere, at distance $r$ from the center. [Use the results in the notes for spherical shells.]
   b. Imagine the planet has a long thin hole along one diameter as shown. A particle of mass $m$ is dropped into the hole at the surface of the planet. What is the force on it when it is at distance $r$ from the center?
   c. What kind of motion will the particle execute?

4. Questions about gravitational potential energy.
   a. For a spherically symmetric object of radius $R$ and mass $M$, show that the gravitational field at the surface is $g = \frac{GM}{R^2}$, directed toward the center.
   b. Use the formula in the notes (page 6) for the potential energy of a point mass $m$ at points outside a spherical mass of radius $R$ and mass $M$. Suppose the point mass is at height $h$ above the surface of the sphere. Show that the potential energy is approximately $U(h) = mgh + \text{const}$. [Use the formula in (a) and the binomial approximation on the formula sheet.]
   c. A mass $m$ is projected with speed $v_0$ upward from the surface of a spherically symmetric planet of mass $M$ and radius $R$. What is its total energy in terms of $m$, $g$, $v_0$ and $R$? [Use the formula from the notes referred to in (b).] Ans: $\frac{1}{2}mv_0^2 - mgR$.
   d. What minimum total energy will let the particle “escape”, that is, not return? What minimum speed $v_0$ must it have at the surface of the planet?

5. Questions about weight.
   a. Explain why astronauts in orbit around the earth are weightless.
   b. Is your weight at the equator slightly less than your weight in London? Explain.
   c. Is your weight at noon, with the sun overhead, slightly less than your weight a midnight, with the sun below your feet? Explain.
6. Questions about tidal forces.
   a. Draw diagrams showing the positions of the earth, sun and moon at a
time when there are spring tides, and again when there are neap tides.
   b. A system of two balls of mass $m$ connected by a
massless rod of length $L$ is falling toward a large
star of mass $M$, as shown. The center of the star is
at distance $r$ from the center of the rod. What are
the forces $F_L$ and $F_R$ exerted by the star on the left and right hand balls?
   c. Find the tension $T$ in the rod that will hold the two ball system together.
   [Draw free body diagrams for each ball.] $Ans: T = \frac{1}{2}(F_L - F_R)$. 
   d. Use the fact that $r >> L$ and the binomial approximation to get an
approximate formula for $T$ in terms of the given quantities. $Ans: \frac{GMmL}{r^3}$.

7. Questions on Kepler’s laws.
   a. Until we were able to send a space probe close to Mercury, its mass was
very poorly known. Why?
   b. Two astronomical objects are observed to orbit around each other in a
circle, with the distance $r$ between them remaining constant. Show that the
ratio $r^3 / T^2$, where $T$ is the period of the rotation, is given by
$G(m_1 + m_2) / 4\pi^2$, where $m_1$ and $m_2$ are the masses of the objects.
   c. A particle moves with constant velocity as shown. Prove
that the line from point $P$ to the particle sweeps out equal
areas in equal times. That is, show that the areas of
successive triangles created by the moving line during
successive equal time intervals $\Delta t$ are all the same.

8. Comet Encke has a perihelion distance of 0.33 AU and a period of 3.3 yr.
   a. What is the semi-major axis $a$ of the orbit? $Ans: 3.32 \times 10^{11}$ m.
   b. What is the aphelion distance for this comet? $Ans: 6.15 \times 10^{11}$ m.
   c. What is its semi-minor axis $b$? $Ans: 1.74 \times 10^{11}$ m.

Use: $GM = 1.33 \times 10^{20}$ in SI units, where $M$ is the sun’s mass; 1 AU = $1.5 \times 10^{11}$ m;
1 yr = $3.16 \times 10^7$ s.
9. A spaceship of mass $m = 10^3$ kg is in a circular orbit of radius $R$ around the earth as shown. Its orbital speed is 4 km/s. [All answers are numerical.]

a. What is its total energy? [Relate $U$ to $K$.] Ans: $-8 \times 10^9$ J.

The engines of the spaceship are fired, sending it into an elliptical orbit with apogee distance $3R$.

b. Sketch the new orbit on the drawing.

c. How much energy was added by the engines? Ans: $4 \times 10^9$ J.

d. When it is at apogee the engines are fired again to put it into a circular orbit of radius $3R$. How much energy must the engines add this time? Ans: $\frac{4}{3} \times 10^9$ J.