Physics 182

Sources of Radiation

Overview.

Maxwell's theory not only explained the nature of e-m waves, including light, but it contained methods for creating those waves. The fundamental source is accelerating charges. However, for visible light and waves of comparable or higher frequencies, it was found that one cannot apply classical mechanics to the charges in their sources (atoms, molecules, or smaller objects), and the theory that must be used (quantum mechanics) does not deal with simple concepts such as acceleration of particles in any simple way. So the statement that accelerating charges produce e-m radiation is true, but not all radiation is produced by *classically* moving charges.

Of course we are dealing with classical physics in this course, so we are interested only in the case where the radiation can be attributed to accelerating charges. Those charges can be those constituting currents in wires, or single particles.

The case of a single accelerating charge is much less simple than one might think, and we do not have the time to deal with its complexities. So we will only treat radiation by charges in currents.

Of course if the charges are accelerating the current cannot be steady. We will treat only the practically important case where it is a sinusoidally alternating current in the form of an oscillating electric dipole. This represents the radiation from radio and TV antennas.

Retarded potentials.

First some general principles. The fields are related to the potentials by

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \ \ \mathbf{B} = \nabla \times \mathbf{A} \,.$$

It is convenient when treating radiation to choose the potentials to obey the condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \,.$$

This is the "Lorentz gauge" condition. Then Gauss's law for **E** gives

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}, \tag{1}$$

while Ampere's law gives

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j}. \tag{2}$$

These are like the wave equation, except that the right side is not zero.

Now we intend to treat cases where the sources (ρ and $\mathbf{j}$) depend on time as well as position. The simplest charge source is a point charge, of course. If we temporarily violate charge conservation and consider a varying point charge at the origin $q(t)$ as the only source around, we find that the solution of Eq (1) is

$$\phi(\mathbf{r},t) = \frac{q(t-r/c)}{4\pi\varepsilon_0}\frac{1}{r}.$$

The proof is in G, but it uses the Dirac delta function to represent the "density" for a point charge.

What this shows is that the field at distance *r* from the source charge at time *t* depends on the strength of the charge at the earlier time $t-r/c$. This is called the *retarded* time, and it represents the fact that information about anything, including e-m fields, cannot travel faster than speed *c*. A test charge at the field point would respond to the field "emitted" by the charge at the origin at a time *earlier* than the present by the amount necessary for the information to travel the distance *r*.

Generalizing this result, we expect the solutions of Eqs (1) and (2) to have the form

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}',t-R/c)}{R}, \tag{3}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{j}(\mathbf{r}',t-R/c)}{R}. \tag{4}$$

Here $R = |\mathbf{r}-\mathbf{r}'|$ as usual.

These are the tools we need to investigate the fields established by an oscillating dipole.

Radiation by an oscillating dipole.

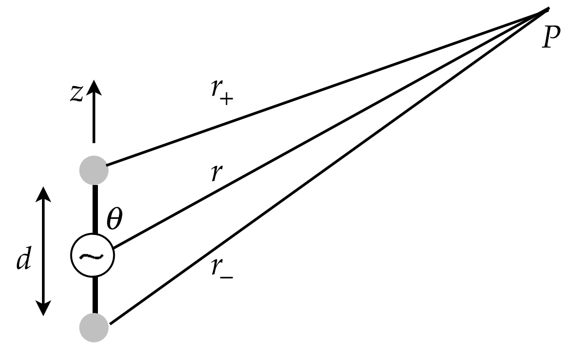
We consider the situation shown, with a generator driving charge to the two small balls. The top ball's charge is given by

$$q(t) = q_0 \sin \omega t \; ;$$

the current running between the balls is

$$I(t) = -\omega q_0 \sin \omega t \; .$$

The dipole moment is $p(t) = q(t)d = p_0 \sin \omega t$, where $p_0 = q_0 d$.



The potential at point P is given by

$$V(\mathbf{r},t) = \frac{q_0}{4\pi\varepsilon_0}\left[\frac{\cos\omega(t - r_+ / c)}{r_+} - \frac{\cos\omega(t - r_- / c)}{r_-}\right]. \tag{5}$$

The vector potential has only a z-component:

$$A_z(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{-d/2}^{+d/2} \frac{-\omega q_0 \sin\omega(t - r / c)}{r} dz \; . \tag{6}$$

We impose three conditions on the distances: $d << c / \omega << r$. Here $c / \omega = \lambda / 2\pi$, where λ is the wavelength of an e-m wave. These conditions allows us to make many approximations, starting with

$$r_\pm = [r^2 + (d/2)^2 \mp rd\cos\theta]^{\frac{1}{2}} \approx r \mp \tfrac{1}{2} d\cos\theta,$$

$$\frac{1}{r_\pm} \approx \frac{1}{r} \pm \frac{d}{2r^2}\cos\theta.$$

Then $\cos\omega(t - r_\pm / c) \approx \cos[(\omega t - \omega r / c) \pm \tfrac{1}{2}(\omega d / c)\cdot\cos\theta]$. Using the formula for the cosine of a sum of angles, then using the fact that $\omega d / c << 1$ and small angle approximations, this becomes $\cos\omega(t - r / c) \mp (\omega d / 2c)\cdot\cos\theta\cdot\sin\omega(t - r / c)$.

Putting these approximations into Eq (5) we find

$$V(r,\theta,t) = \frac{p_0\cos\theta}{4\pi\varepsilon_0}\left[\frac{\cos\omega(t - r / c)}{r^2} - \frac{\omega\sin\omega(t - r / c)}{cr}\right].$$

For a static dipole ($\omega = 0$) we have only the first term, which is the usual potential for a dipole. This term in $V(r,\theta,t)$ gives the potential of the *near field*, basically like the static field except dependent on time.

The second term, which falls off only as $1/r$, gives the radiation field. Since our interest is in radiation, and since for large *r* this is the dominant term, we write

$$V_{rad}(r,\theta,t) = -\frac{p_0\omega}{4\pi\varepsilon_0 c}\frac{\cos\theta}{r}\sin\omega(t - r/c)\,. \tag{7}$$

The vector potential is easier. First, the integral in Eq (6) runs only over the small distance *d*, so we approximate the integrand by its value at the origin:

$$(A_{rad})_z(r,\theta,t) \approx -\frac{\mu_0 p_0\omega}{4\pi}\frac{\sin\omega(t - r/c)}{r}\,. \tag{8}$$

From these potentials we calculate **E**, **B** and **S**.

First $\mathbf{E} = -\nabla V - \partial\mathbf{A}/\partial t$. Since V depends only on *r* and θ, the gradient has only those components:

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}}\,.$$

Carrying out the derivatives and dropping terms in $1/r^2$, we find

$$\nabla V = \frac{p_0\omega^2}{4\pi\varepsilon_0 c^2}\frac{\cos\theta}{r}\cos\omega(t - r/c)\cdot\hat{\mathbf{r}}\,.$$

To deal with **A** we note that the relation between cartesian and spherical coordinates gives $\hat{\mathbf{k}} = \cos\theta\cdot\hat{\mathbf{r}} - \sin\theta\cdot\hat{\boldsymbol{\theta}}$, so in those terms

$$\mathbf{A}_{rad}(r,\theta,t) \approx -\frac{\mu_0 p_0\omega}{4\pi}\frac{\sin\omega(t - r/c)}{r}\cdot(\cos\theta\cdot\hat{\mathbf{r}} - \sin\theta\cdot\hat{\boldsymbol{\theta}})\,.$$

Then we have

$$\frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi}\frac{\cos\omega(t - r/c)}{r}(\cos\theta\cdot\hat{\mathbf{r}} - \sin\theta\cdot\hat{\boldsymbol{\theta}})\,.$$

Using $\mu_0 = 1/\varepsilon_0 c^2$, the terms proportional to $\hat{\mathbf{r}}$ cancel when we add the two contributions to **E**, and we have

$$\mathbf{E}_{rad} = -\frac{p_0\omega^2}{4\pi\varepsilon_0 c^2}\frac{\sin\theta}{r}\cos\omega(t - r/c)\cdot\hat{\boldsymbol{\theta}}\,. \tag{9}$$

The lines of this field run clockwise around lines directly outward from the dipole.

From $\mathbf{B} = \nabla \times \mathbf{A}$ we have only a ϕ - component (see the cover of G for the curl):

$$\mathbf{B} = \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} + \frac{\partial A_r}{\partial \theta}\right] \cdot \hat{\boldsymbol{\phi}}.$$

Since $A_r \sim 1/r$ the last term will give a result $\sim 1/r^2$, which we neglect. The remaining term is

$$\mathbf{B}_{rad} = -\frac{p_0 \omega^2}{3\pi\varepsilon_0 c^3} \frac{\sin\theta}{r} \cos\omega(t - r/c) \cdot \hat{\boldsymbol{\phi}}. \tag{10}$$

This field's lines make circles around the line along the dipole.

Note that the fields in Eqs (9) and (10) satisfy the general properties of an e-m wave:

1. They are mutually perpendicular, and both perpendicular to the vector from the source to the field point.
2. They oscillate exactly in phase.
3. The magnitudes obey $B = E/c$.
4. Both magnitudes decrease with distance as $1/r$.

Now we find the Poynting vector. Instantaneously it is

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} = \frac{p_0^2 \omega^4}{16\pi^2 \varepsilon_0 c^3} \frac{\sin^2\theta}{r^2} \cos^2 \omega(t - r/c) \cdot \hat{\mathbf{r}}.$$

Averaging of a cycle turns the $\cos^2$ into $1/2$, so we have

$$\mathbf{S}_{av} = \frac{p_0^2 \omega^4}{32\pi^2 \varepsilon_0 c^3} \frac{\sin^2\theta}{r^2} \cdot \hat{\mathbf{r}}. \tag{11}$$

Note the dependence on $\sin^2\theta$. There is no radiation along the line of the dipole, and the maximum intensity is in directions perpendicular to it. Also note the factor ω^4, which makes the power radiated rise rapidly with the frequency of oscillation.

One can get the total power radiated in all directions by integrating the flux of **S** over a large sphere of radius *r*. The result is Larmor's formula:

$$P_{av} = \frac{p_0^2 \omega^4}{12\pi\varepsilon_0 c^3}.$$

Omitted cases.

We have treated in detail only the case of an oscillating electric dipole. This is a very important case, but it hardly covers all of the mechanisms of e-m radiation. The others are covered in G, including the general case for a continuous distribution and currents. Probably the most important omitted case is that of a single accelerated charge. Here we give a brief account without proofs of the radiation in that case, considering only motion along a straight line.

The potentials for a single moving charge are called Liénard-Wiechert potentials, and exhibit features of the problem that almost cry out to be treated by the methods of relativity. (This gives further evidence that electromagnetism is an inherently relativistic theory.) If the velocity of the charge is small compared to c the radiation is very similar to that of the dipole we discussed, leading to the formula for **S**:

$$\mathbf{S} = \frac{q^2 a^2}{16\pi^2 \varepsilon_0 c^3} \frac{\sin^2\theta}{R^2} \hat{\mathbf{R}} .$$

Here θ is the angle between the acceleration vector **a** and **R**, where the latter is the vector from the charge to the field point at the *retarded* time. The pattern in terms of θ is like that of the dipole: there is no radiation in the direction of **a**, and the maximum intensity is perpendicular to **a**.

The general case is more complicated, but the formula that replaces Larmor's is relatively simple:

$$P = \frac{q^2}{6\pi\varepsilon_0 c^3} \gamma^6 \left[a^2 - \left(\frac{\mathbf{v} \times \mathbf{a}}{c} \right)^2 \right].$$

Here $\gamma = 1/\sqrt{1-(v/c)^2}$ is the usual relativistic factor. For the simple case where the motion is along a straight line the last term in [ ] is zero, of course. In that case the main new feature is the appearance of the factor γ^6. This greatly increases the power radiated as the speed approaches c.

A common example of this kind of radiation is *bremsstrahlung*, the radiation that occurs when a charged particle is rapidly stopped, as in the target of an X-ray tube.