Physics 182

Electromagnetic Waves

Overview.

The most remarkable conclusion of Maxwell's work on electromagnetism in the 1860's was that waves could exist in the fields themselves, traveling with the speed of light. That light has wave properties was well established by then, and its speed was measured to quite high precision in the 1850's. Its polarization properties had been shown by Fresnel and others to be consequences of the fact that it is a transverse wave. But nothing before Maxwell had connected light to electromagnetism.

In these notes we will give a modern proof of the existence of these waves and their properties, derived from the field equations alone. In the next set of notes we will discuss how such waves can be generated by electromagnetic means. That does not apply to light, however, since an account of how atoms and molecules emit light necessarily involves quantum theory.

The wave equation.

We start from the vector identity $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$ and the field equations in empty space (no charges or currents):

$$\nabla\cdot\mathbf{E} = 0\,,\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\,;$$

$$\nabla\cdot\mathbf{B} = 0\,,\ \nabla\times\mathbf{B} = \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\,.$$

Applying the identity to $\mathbf{E}$ we find

$$-\nabla\times\frac{\partial\mathbf{B}}{\partial t} = -\nabla^2\mathbf{E}\,.$$

But, interchanging the space an time partial derivatives, we have

$$\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) = \mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}\,,$$

so we have

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

This is the classical wave equation, and the speed of the waves is given by $v^2 = 1/\mu_0\varepsilon_0$. Now in fact, numerically, $1/\mu_0\varepsilon_0 = c^2$, where c is the measured speed of light. So there is an E-field obeying Maxwell's equations that also obeys the wave equation for waves with speed c.

Similar manipulation using **B** gives the same equation:

$$\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

Of course not all possible fields are those of waves. A constant and uniform E-field would certainly satisfy the wave equation, but it hardly qualifies as a wave. And not all vector fields that obey the wave equation also obey Maxwell's equations. What has been shown is that there are field configurations that are waves.

Consider the case of a wave traveling only in the $+x$-direction. Its dependence on x and t must be of the form $f(x-ct)$. We can write the E-field of such as wave as

$$\mathbf{E} = \mathbf{n} f(x-ct),$$

where **n** is a unit vector giving the direction of **E**. But the field must also obey $\nabla \cdot \mathbf{E} = 0$. We find $n_x \cdot f'(x-ct) = 0$, and since f' is not always zero this shows that **n** is perpendicular to x. That is, the field is perpendicular to the direction of wave propagation. It is a **transverse** wave. The same argument shows that the B-field is also perpendicular to the direction of wave propagation.

How are the two fields related? We use the other Maxwell equations. For example, let the E-vector be in the y-direction: $E_y = f(x-ct)$. Then $\nabla \times \mathbf{E}$ has only a z-component: $(\nabla \times \mathbf{E})_z = \partial_x E_y = f'(x-ct)$. So Faraday's law reads $\partial_x E_y = -\partial_t B_z$, which gives

$$-\partial_t B_z = f'(x-ct).$$

We can integrate this to find $B_z = (1/c) f(x-ct)$ — plus a constant which we ignore.

We have shown three things:

- **E** and **B** are perpendicular with $\mathbf{E} \times \mathbf{B}$ in the direction of wave propagation;
- the magnitudes obey $E = cB$;
- the dependence on x and t is the same for both fields.

The first of these is clear from the meaning of the Poynting vector, which gives both the direction of energy flow and its intensity. The second means that the energy is carried equally by the two fields, since if $E = cB$ the electric and magnetic energy densities are equal. The third means that the fields propagate exactly in phase with each other.

If we had started this analysis with the B-field, we would have used Ampere's law to connect the fields, but the result would have been the same.

Harmonic waves in 3-D

An important category of solutions of the wave equation is the harmonic functions (especially sines and cosines). We use the Euler trick, replacing a cosine by the real part of a complex exponential. A wave with wavelength λ and frequency f is described by

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

where $\mathbf{k}$ is in the direction of propagation, with magnitude $k = 2\pi/\lambda$, and $\omega = 2\pi f$. Applying $\nabla \cdot \mathbf{E} = 0$ to this function, we find as expected $\mathbf{k}\cdot\mathbf{E} = 0$, showing the transversality of the wave.

Exercise: Carry out the derivatives to show this.

For $\nabla \times \mathbf{E}$ we find $\mathbf{k}\times\mathbf{E}$, so Faraday's law gives $\mathbf{k}\times\mathbf{E} = i\omega\mathbf{B}$. This shows that **B** is perpendicular to both **k** and **E**, and the magnitude equation shows that $E = cB$ (since $\omega/k = c$).

The product $\mathbf{E}\times\mathbf{B} = \dfrac{1}{i\omega}\mathbf{E}\times(\mathbf{k}\times\mathbf{E}) = \dfrac{\mathbf{k}}{i\omega}E^2$ shows that the direction of propagation is the same as the direction of $\mathbf{E}\times\mathbf{B}$. The magnitude gives $|\mathbf{E}\times\mathbf{B}| = E^2/c$. Using the definition of the Poynting vector we wee that $S = E^2/\mu_0 c = \varepsilon_0 c E^2$, a useful formula for the intensity of the wave.

Effects of dielectric media

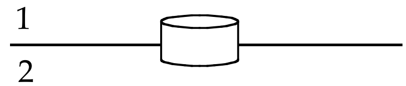
If the wave is traveling in a transparent dielectric medium, two of the field equations are different: $\nabla\cdot(\kappa\mathbf{E}) = 0$ and $\nabla\times\mathbf{B} = \mu_0\varepsilon_0 \cdot \partial(\kappa\mathbf{E})/\partial t$. Retracing the steps we used in deriving the wave equation, we find (assuming κ is a constant, as in a simple dielectric)

$$\nabla^2\mathbf{E} = \mu_0\varepsilon_0\kappa\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

This means the speed of the waves is different: $v = c / \sqrt{\kappa}$. This shows that the index of refraction, defined by $n = c / v$, is simply $\sqrt{\kappa}$.

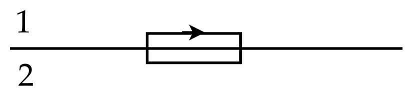
As long as awe stay in the same medium, this is the only difference. But if the wave crosses an interface between two media with different values of κ, then the equation $\nabla \cdot (\kappa \mathbf{E}) = 0$ has new consequences. It is easier to handle in the integral form:

$$\oint (\kappa \mathbf{E}) \cdot d\mathbf{A} = 0.$$



Shown is the boundary between two such media. Let the E-field be directed from medium 1 into medium 2. We use the pillbox surface shown to evaluate the flux integral. The flux through the top is $-A \cdot \kappa_1 E_{1\perp}$, where A is the area of the top and $E_{1\perp}$ is the component of **E** normal to the surface of the top. (The negative sign comes from our assumption that the field is directed downward.) For the bottom we have flux $+A \cdot \kappa_2 E_{2\perp}$. We shrink the height of the pillbox to zero, so the sides do not contribute. We find, setting the total flux to zero, $\kappa_1 E_{1\perp} = \kappa_2 E_{2\perp}$. That is:

The component of $\kappa \mathbf{E}$ normal to the interface is continuous.



The other equation involving the E-field across a boundary is Faraday's law, which in integral form is $\oint \mathbf{E} \cdot d\mathbf{r} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$. We use the rectangular path shown.

Call the component of **E** parallel to the interface and to the right $E_{\|}$. Then the horizontal part of the path above the interface gives $E_{1\|} \cdot \ell$, where ℓ is the width of the rectangle. The horizontal path below the interface gives $-E_{2\|} \cdot \ell$. Again we shrink the height to zero so the area goes to zero and the flux on the right side of Faraday's law disappears. We find that $E_{1\|} = E_{2\|}$, or:

The component of **E** parallel to the interface is continuous.

These are the boundary conditions we will use at such interfaces.

The B-field is continuous in both components unless there are magnetic properties of the media, which we neglect because they have very little effect unless the media are ferromagnetic.

The other general principle governing the propagation of waves across such an interface is conservation of energy. These three principles completely determine what happens with reflection and refraction of e-m waves.

Reflection and refraction.

When a wave impinges on an interface between two transparent media, it divides into two waves, one reflected back into the original medium, one transmitted into the other medium. We write the E-fields of the waves as follows:

$$\text{Incident: } \mathbf{E}_i = \mathbf{E}_1 e^{i(\mathbf{k}_1\cdot\mathbf{r}-\omega t)}$$
$$\text{Reflected: } \mathbf{E}_r = \mathbf{E}_1' e^{i(\mathbf{k}_1'\cdot\mathbf{r}-\omega t)}$$
$$\text{Transmitted: } \mathbf{E}_t = \mathbf{E}_2 e^{i(\mathbf{k}_2\cdot\mathbf{r}-\omega t)}$$

The frequency is unchanged, so ω is the same in all three functions. The wavelength, and therefore the magnitude of k is the same for both waves in the original medium. As always, the wave speed is $v = \omega / k$.

The *total* field in medium 1 is $\mathbf{E}_i + \mathbf{E}_r$, and it is to this field that the boundary conditions above apply. We choose a coordinate system with the interface as the x-z plane and with the incident wave vector $\mathbf{k}_1$ in the x-y plane; we also choose the point where the wave strikes the interface to have $y = 0$.

Our boundary conditions are these. At $y = 0$:

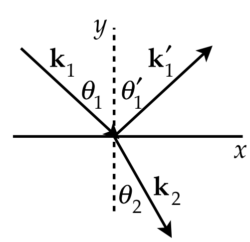
$$\mathbf{E}_{\|} \text{ is continuous: } (E_1)_{x,z} e^{i\mathbf{k}_1\cdot\mathbf{r}} + (E_1')_{x,z} e^{i\mathbf{k}_1'\cdot\mathbf{r}} = (E_2)_{x,z} e^{i\mathbf{k}_2\cdot\mathbf{r}}, \qquad (1)$$

$$\kappa\mathbf{E}_{\perp} \text{ is continuous: } (\kappa_1 E_1)_y e^{i\mathbf{k}_1\cdot\mathbf{r}} + (\kappa_1 E_1')_y e^{i\mathbf{k}_1'\cdot\mathbf{r}} = (\kappa_2 E_2)_y e^{i\mathbf{k}_2\cdot\mathbf{r}}. \qquad (2)$$

$$\mathbf{B} = (\mathbf{k}\times\mathbf{E})/\omega \text{ is continuous (both components).} \qquad (3)$$

These must hold for all values of x and z, so the x-components of all the $\mathbf{k}$'s are equal, and so are the z-components. Since $\mathbf{k}_1$ has no z-component, all the z-components must be zero. This means all the $\mathbf{k}$ vectors lie in the x-y plane, which is called the *plane of incidence*.

Shown is the plane of incidence, with the $\mathbf{k}$ vectors and the interface. That the x-components are all equal implies that $k_1 \sin\theta_1 = k_1' \sin\theta_1'$, and $k_1 \sin\theta_1 = k_2 \sin\theta_2$.



Since $k_1 = k_1'$, the first of these gives a fundamental law of waves:

$$\text{Law of reflection: } \theta_1 = \theta_1'.$$

Since $k_1 / k_2 = v_2 / v_1$ the second gives (using $n = c / v$) another fundamental law:

Law of refraction: $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

Now we return to Eqs (1-3) with all the exponentials having canceled. We distinguish two cases: (a) **E** lies in the plane of incidence ($E_z = 0$); (b) **E** is perpendicular to that plane ($E_x = E_y = 0$).

(a) **E** in plane of incidence. In this case **B** is normal to that plane. Then Eq (1) gives $(E_1 + E_1')\cos\theta_1 = E_2\cos\theta_2$, and Eq (2) gives $\kappa_1(E_1 - E_1')\sin\theta_1 = \kappa_2 E_2 \sin\theta_2$. Remembering that $\kappa = n^2$ and the law of refraction, the second of these is $n_1(E_1 - E_1') = n_2 E_2$. (Continuity of **B** gives the same equation, since $B = E/v$.)

We define

$$\alpha = \cos\theta_2 / \cos\theta_1, \ \beta = n_2 / n_1.$$

The two equations become

$$E_1 + E_1' = \alpha E_2, \ E_1 - E_1' = \beta E_2.$$

We find from these

$$\frac{E_1'}{E_1} = \frac{\alpha - \beta}{\alpha + \beta}, \ \frac{E_2}{E_1} = \frac{2}{\alpha + \beta}. \tag{4}$$

Now we look at the transport of energy. The power delivered per unit area to the interface by the incident wave is the component of the Poynting vector normal to the surface, which is $S_1 \cos\theta_1$, where $S_1 = \frac{1}{\mu_0} E_1 B_1 = \frac{1}{\mu_0} E_1 \frac{E_1}{v_1}$, since in a medium $B = E/v$. Using $\frac{1}{\mu_0} = \varepsilon_0 c^2$ and $n_1 = c/v_1$, we have $S_1 = \varepsilon_0 c n_1 E_1^2$. For the reflected wave we have, similarly, $S_1' = \varepsilon_0 c n_1 E_1'^2$. The ratio of the reflected intensity to the incident intensity is the *reflectivity*:

$$R = \frac{S_1 \cos\theta_1}{S_1' \cos\theta_1'} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2. \tag{5}$$

For the transmitted intensity we have $S_2 \cos\theta_2 = \varepsilon_0 c n_2 E_2^2 \cos\theta_2$. The fraction of the incident power that is transmitted to the second medium is

$$T = \frac{S_2 \cos\theta_2}{S_1 \cos\theta_1} = \frac{E_2^2}{E_1^2} \frac{n_2}{n_1} \frac{\cos\theta_2}{\cos\theta_1} = \frac{4\alpha\beta}{(\alpha + \beta)^2}. \tag{6}$$

It is easy to see that $R+T=1$, expressing conservation of energy.

There are some special features of reflection that can be seen from Eqs (4-6):

1. Phase change on reflection. Note that, using the law of refraction, we can write
$$\alpha-\beta=\frac{\cos\theta_2}{\cos\theta_1}-\frac{\sin\theta_1}{\sin\theta_2}=\frac{1}{2\sin\theta_2\cos\theta_1}[\sin 2\theta_2-\sin 2\theta_1].$$
This means that if $n_1<n_2$, so that $\theta_2<\theta_1$, we have $\alpha-\beta<0$. Eq (4) then says that E_1'/E_1 is negative. What this means is that the direction of **E** for the reflected wave is opposite to that of the incident wave; the reflection has "flipped" the direction of **E**. This is usually called a sudden phase change of π. On the other hand, if $n_1>n_2$, then $\theta_2>\theta_1$, so $\alpha-\beta>0$ and there is no "flipping" of the wave. This rule is important in interference phenomena.

   It is a general property of waves, not just e-m waves.

2. Reflection at normal incidence. If $\theta_1=0$, then also $\theta_2=0$. This gives $\alpha=1$, and Eq (5) says the reflectivity is simple: $R=\left(\dfrac{n_1-n_2}{n_1+n_2}\right)^2$.

3. Brewster's angle. If $\alpha=\beta$ we have no reflected wave at all (for **E** in the plane of incidence). This condition, together with $n_2/n_1=\sin\theta_1/\sin\theta_2$, leads to $\sin 2\theta_1=\sin 2\theta_2$. The solution $\theta_1=\theta_2$ is not allowed because it requires $n_1=n_2$. The other solution is $\theta_1=\frac{1}{2}\pi-\theta_2$. This is the *Brewster angle* θ_B. The condition for it is usually written as $\tan\theta_B=n_2/n_1$, which follows from the law of refraction. At this angle of incidence there is no wave reflected if **E** is in the plane of incidence. This is not true if **E** is normal to that plane, as we will see. If the incident waves are a mixture of both directions of **E** (i.e., both *polarizations*) then only the waves with **E** normal to the plane will be reflected. This results in a *completely polarized* reflected intensity.

4. Total reflection. If $n_1>n_2$ there will be an angle of incidence such that $\theta_2=\frac{1}{2}\pi$. This makes $\alpha=0$ and therefore $R=1$, meaning there is only a reflected wave, not a transmitted one. This angle is called the critical angle, defined by $\sin\theta_C=n_2/n_1$. For angles of incidence equal or greater than θ_C there is total reflection.

(b) <u>**E** normal to the plane of incidence.</u> This case is simpler because there is no component of **E** normal to the interface. We have simply $E_1 + E_1' = E_2$. Continuity of $\mathbf{B}_\perp$ gives the same equation. But continuity of $\mathbf{B}_{||}$ now says $n_1(E_1 - E_1')\cos\theta_1 = n_2 E_2 \cos\theta_2$. With the same definitions as before we find

$$\frac{E_1'}{E_1} - \frac{1-\alpha\beta}{1+\alpha\beta},\ \frac{E_2}{E_1} = \frac{2}{1+\alpha\beta}. \tag{7}$$

This leads to

$$R = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2,\ T = \frac{4\alpha\beta}{(1+\alpha\beta)^2}. \tag{8}$$

Since α and β are either both less than 1 or both greater than 1, there is no angle for which R vanishes; that is, there is no Brewster angle in this case. There is total reflection, with the same critical angle as before. And if $n_1 < n_2$ then $\alpha\beta > 1$ and reflection causes a phase change of π as before, while for $n_1 > n_2$ there is no phase change.

Using the law of refraction and some trigonometry, we can write the reflectivities in a useful form:

$$\text{For } \mathbf{E} \text{ in the plane of incidence: } R = \left(\frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}\right)^2;$$

$$\text{For } \mathbf{E} \text{ normal to the plane of incidence: } R = \left(\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)}\right)^2.$$

Plots of R vs θ_1 are given in the notes for PHY 54 at

http://www.phy.duke.edu/~lee/P54/home.htm

(Follow the link to "Light" and look on page 9.)

Conductors.

If the wave impinges on a conductor, our assumption that there are no free charges fails. The field equations must also be supplemented by the relation between current density and E-field in the conductor, $\mathbf{j} = \sigma \mathbf{E}$. (Here σ is the conductivity, which we use rather than resistivity to avoid having the symbol ρ appear with two meanings.) We have

$$\varepsilon_0 \nabla \cdot (\kappa \mathbf{E}) = \rho, \ \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t, \ \nabla \times \mathbf{B} = \mu_0 \left(\sigma \mathbf{E} + \varepsilon_0 \kappa \cdot \partial \mathbf{E} / \partial t \right).$$

(Again we have neglected magnetic properties of the materials.) Taking the divergence of the 4th equation and substituting into the 1st equation, we find

$$\varepsilon_0 \kappa \cdot \partial \rho / \partial t + \sigma \cdot \rho = 0.$$

The solutions of this are exponentials in t that die away. We will assume the time for this to happen is short, so the free charge density is negligible. Then using the same manipulations that led to the wave equation, we find

$$\nabla^2 \mathbf{E} = \frac{\kappa}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t},$$

and the same for **B**. Let the wave be moving in the x-direction. We try a solution of the form $\mathbf{E}(x,t) = \mathbf{E}_0 e^{i(kx - \omega t)}$, where we expect k to be complex. Substituting, we find

$$k^2 = \frac{\kappa}{c^2} \omega^2 + i \mu_0 \sigma \omega.$$

Let $z = (kc / n\omega)$, where we have used $\kappa = n^2$. Then we have $z^2 = 1 + i(\sigma / \kappa \varepsilon_0 \omega) = 1 + ia$.

Now let $z = x + iy$. We find $x^2 - y^2 + 2ixy = 1 + ia$, so $y = a / 2x$ and $x^2 - \dfrac{a^2}{4x^2} = 1$, or

$x^4 - x^2 - a^2 / 4 = 0$. This has solution $x^2 = \frac{1}{2}\left(1 \pm \sqrt{1 + a^2}\right)$, in which we take the positive root. Thus the real part of k is

$$\operatorname{Re} k = \frac{n\omega}{c} \frac{1}{\sqrt{2}} \left[1 + \sqrt{1 + (\sigma / \kappa \varepsilon_0 \omega)^2} \right]^{1/2}.$$

We have $y^2 = a^2 / 4x^2$. Rationalizing the fraction we find $y^2 = \frac{1}{2}\left(\sqrt{1 + a^2} - 1\right)$. The imaginary part of k is thus

$$\operatorname{Im} k = \frac{n\omega}{c}\frac{1}{\sqrt{2}}\left[\sqrt{1+(\sigma/\kappa\varepsilon_0\omega)^2}-1\right]^{1/2}.$$

The field dies away in the conductor like $e^{-\operatorname{Im} k\cdot x}$. The distance it goes before falling to $1/e$ of its original value is called the *skin depth*: $d = 1/\operatorname{Im} k$.

We examine two cases:

1. Poor conductor. Then $\sigma/\kappa\varepsilon_0\omega << 1$. Using the binomial approximation on the square root, we find $\operatorname{Im} k \approx \sigma/2n\varepsilon_0 c$. The skin depth is proportional to $1/\sigma$, independent of the frequency. The real part of k becomes $n\omega/c$, as in the case of the previous section.
2. Good conductor. Then $\sigma/\kappa\varepsilon_0\omega >> 1$. We have $\operatorname{Re} k \approx \operatorname{Im} k \approx \sqrt{\mu_0\sigma\omega/2}$. The skin depth decreases with increasing frequency.

Because of the dependence on ω, a material can be a good conductor to e-m waves for low frequencies and a poor one for high frequencies.

Since for a good conductor the E-field is essentially zero inside the material, the boundary conditions tell us that the reflected wave has essential the same intensity as the incident wave, i.e., the reflectivity is essentially 1. There is a phase change on reflection, however.