Physics 182

Energy and Momentum in the Fields

Overview.

Except for the description of magnetic properties of materials — which we omit, partly because of time constraints and partly because an honest description requires quantum theory — we now have all the fundamental principles and equations of classical electrodynamics. In the period from about 1860, when it was first published, to 1900 the adherents to Maxwell's theory worked out its consequences. This period coincided with the "second industrial revolution" brought about by the introduction of devices using electric power. Electronics came later, but the rules governing circuits of various types were worked out, based on the relatively new understanding of conservation of energy.

It was only in 1887, after Maxwell's death, that direct confirmation was obtained for his prediction of electromagnetic radiation. The ability to send signals through empty space by "aether waves" led quickly to wireless telegraphy and eventually to radio. The theoretical description of radiation will be discussed in the next set of notes.

But a signal implies transfer of energy and momentum. How the fields carry those quantities is the subject of these notes.

Poynting's theorem.

Consider the quantity $\nabla\cdot(\mathbf{E}\times\mathbf{B})$. By a vector calculus identity (see the front cover of G) it is equal to $\mathbf{B}\cdot\nabla\times\mathbf{E}-\mathbf{E}\cdot\nabla\times B$. Using Faraday's law and Ampere's law we have

$$\nabla\cdot(\mathbf{E}\times\mathbf{B}) = -\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}-\mu_0\varepsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}-\mu_0\mathbf{E}\cdot\mathbf{j}. \qquad (1)$$

Now the total energy density of the fields is

$$\eta_{em} = \tfrac{1}{2}\varepsilon_0 E^2 + \tfrac{1}{2\mu_0}B^2,$$

and its time derivative is

$$\frac{\partial\eta_{em}}{\partial t} = \varepsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}+\frac{1}{\mu_0}\frac{\partial\mathbf{B}}{\partial t}.$$

Dividing Eq (1) by μ_0 and rearranging we find

$$\nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) + \mathbf{E} \cdot \mathbf{j} + \frac{\partial \eta_{em}}{\partial t} = 0 \tag{2}$$

To interpret this we integrate it over a volume and use the divergence theorem:

$$\int_V \frac{\partial \eta_{em}}{\partial t} d^3r + \int_V (\mathbf{E} \cdot \mathbf{j}) d^3r + \oint \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) \cdot d\mathbf{A} = 0 \,. \tag{3}$$

The first term represents the change in field energy within the volume. The other two terms must therefore represent ways in which that energy is changed. The second term represents the work done on moving charges by the E-field; this could be many kinds of things, from accelerating charges in X-ray machines to running electric motors, or simply Joule heating. Those are changes taking place within the volume. The last term must represent flow of energy in or out through the surface surrounding the volume. Eq (3) is a statement of conservation of energy: the rate of change of field energy in the volume is balanced by the rate at which field energy is transferred to (or from) other forms and by the rate at which energy flows into (or out of) the volume. Eq (2) and Eq (3) are statements of what is called Poynting's theorem.

The integrand in the last term in Eq (3) is called the **Poynting vecto**r:

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \,.$$

Its direction gives the direction of flow of energy in the fields, and its magnitude is the amount of such flow per unit area per unit time. The latter is called the *intensity*.

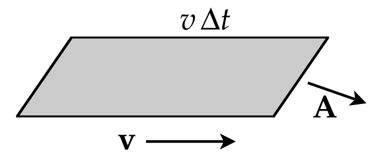
The really new thing in Poynting's theorem is the understanding of how energy moves entirely by means of the fields themselves. This is a very important insight in the case of electromagnetic radiation, but its usefulness is not restricted to that. We will see some examples in the assignment.

Momentum in the fields.

As can be intuitively understood, if there is flow of energy through a surface there is likely to be momentum involved as well. The relation between the two is a very general law. We will derive it from the case where the energy and momentum are those of a stream of particles, but we must be careful not to use the Newtonian approximation because we want it also to apply to electromagnetic energy flow (which travels at the speed of light).

Let there be a stream of particles, all moving with velocity **v**. We consider the flow through a cylinder as shown below. The area of the ends is not perpendicular to the sides, but is in the direction of **A**. The length of the cylinder is the distance the particles travel in time Δt. Let the energy flow vector be **S** (not necessarily the Poynting vector). Then the energy flowing out of the cylinder in time Δt is $\mathbf{S} \cdot \mathbf{A}\,\Delta t$.

This is also the energy of the particles in the volume of the cylinder at any instant, since in time Δt all the particles in that volume flow out. Let the number of particles per unit volume be n and let each have total energy (given by the relativistic formula) $E = m\gamma c^2$, where $\gamma = (1 - v^2 / c^2)^{-1/2}$.



This formula includes the rest energy mc^2 as part of the energy. We have $E = mc^2 + mc^2(\gamma - 1)$, where the second term is the kinetic energy. For $v / c << 1$ the latter reduces to the Newtonian formula $\frac{1}{2}mv^2$.

The total energy of the particles that flows out in time Δt is thus $n \cdot m\gamma c^2 \cdot (\mathbf{v} \cdot \mathbf{A})\Delta t$. Comparing these two formulas for the total energy flow we have $\mathbf{S} = n \cdot m\gamma c^2 \mathbf{v}$. Now the relativistic formula for the momentum of a particle is $\mathbf{p} = m\gamma \mathbf{v}$, so we see that $\mathbf{S} = c^2 \cdot n \cdot \mathbf{p}$. Finally, we define the **momentum density g** to the be total momentum per unit volume, which in this case is just $\mathbf{g} = n \cdot \mathbf{p}$. Then we have finally $\mathbf{S} = c^2 \mathbf{g}$, or

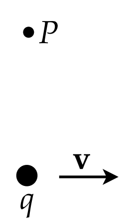
$$\text{Energy flux} = c^2 \cdot \text{Momentum density}.$$

This general formula applies to energy carried by the electromagnetic field as well as to material objects, so a flow of energy in the fields implies momentum carried by the fields as well.

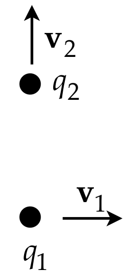
For electromagnetic radiation we have a simple relation between the energy flux magnitude and the energy density in the fields: $S = c \cdot \eta_{em}$. Thus in that case $\text{Energy density} = c \cdot \text{Momentum density}$. This is an important property of e-m radiation.

Two examples.

Consider a positive charged particle moving as shown. We wish to find the directions of the fields at the point indicated, which is directly above the charge. The E-field is directly away from the charge, therefore vertical. The B-field direction can be obtained by treating the moving charge as a bit of current in the direction of **v**; the Biot-Savart law gives the direction of **B** as out of the page. The Poynting vector is to the right, parallel to **v**. This shows that the energy in the fields of the charge move with it, as one would expect.



The second case involves the force between two moving point charges, both positive and moving as shown at a particular instant. The electric (Coulomb) forces between the two are equal and opposite, obeying the 3rd law. But consider the magnetic forces. The B-field created by q_1 at the location of q_2 is out of the page, so the force exerted on q_2 by q_1 is to the right. The B-field created by q_2 at the location of q_1, however, is zero (because q_2 is moving directly away from q_1). The magnetic forces are not equal and opposite, indicating a violation of the 3rd law.



Now the important content of the 3rd law is conservation of momentum, and what this example shows is that the total momentum of the two particle system is *not* conserved.

The resolution of this problem involves the energy and momentum in the fields themselves. There is a flow of field energy going on here, and therefore momentum is in the fields. The "missing" momentum of the particles can be accounted for if we include the fields as part of the mechanical system. A more detailed analysis requires expressing the energy and momentum of the system in a (four-dimensional) second rank tensor, so we will not present it here.

The major point of all this, however, is that isolated systems involving interaction through electromagnetic fields must include the energy in the fields in order that total energy and momentum of the system be conserved. A secondary point is that to do the analysis properly one must abandon the Newtonian approximation and use the correct relativistic formulas for the energy and momentum of particles.