Physics 182

Electrodynamics

Overview.

So far the fields we have been studying have been independent of time, and the field equations for **E** do not contain **B**, or vice versa. Electricity and magnetism seem to be related only through the fact that currents create magnetic fields. Scientists in 1830 suspected that a current was a flow of electric charge, but they had no direct evidence as to what the charges in a wire might be.

The separation of the two fields of study was ended by experiments in 1830-1. In what may be the most important finding in terms of practical use of energy since the taming of fire, Faraday showed (in effect) that a wire in a changing magnetic field will carry a current just as though it had a small battery in it. The E-field that produces this current is called *induced*, and the process is *electromagnetic induction*. The essence of the matter is that a changing magnetic field produces an electric field.

The induced E-field is not like the one created by charges. Its field lines close on themselves; it is not related to the scalar potential V but rather to the vector potential **A**; its strength falls off with distance from the source only as $1/r$.

In the 1840's attention turned to electric circuits, where the work of Ohm and Kirchhoff was instrumental. The rules relating resistance to current involve the concept of *electromotive force* (emf), which is the work per unit charge that sets currents into motion. Among methods of producing an emf is the induced E-field arising from changing magnetic fields. This led quickly to the invention of electromagnetic generators and to the "second industrial revolution" of the late 1800's.

Meanwhile various versions of a theoretical framework to encompass these new results were offered. In 1860 Maxwell produced his famous equations, showed that light is an electromagnetic wave, and paved the way for wireless communication. In the process he reasoned theoretically (without direct experimental proof) that, just as a changing B-field produces and E-field, so the converse is true: a changing E-field produces a B-field. Maxwell showed that this must be so if one is to have the law of charge conservation.

In basic terms the theory of electromagnetism was now complete. For the next 50 years Maxwell and his successors worked out the consequences, especially in terms of radiated energy. Those will be covered in later units.

Circuits and emf.

As soon as researchers had "Voltaic piles" (as batteries were called then) they started investigating the properties of what we now call DC circuits. But it took a long time (because of the lack of good measuring devices, mainly) to establish the relationships between current, resistance and emf. What is called Ohm's law was not generally accepted until nearly 1850, and the Kirchhoff rules for circuits were published about that time. It was not until the concept of energy began to be applied to electricity that things became clear.

There is an excellent account of these things in G. Here is a summary:

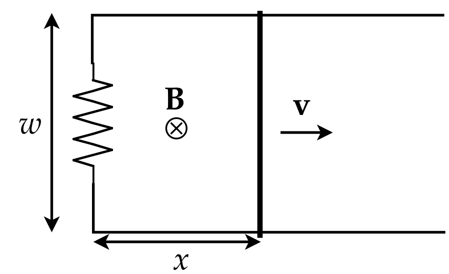
- When a source of emf (i.e., something external that does work on the charges) is applied to a circuit, a current is quite rapidly established in all parts of the circuit. If the emf is time-independent (e.g., a battery) the current reaches a steady value very quickly. Its magnitude is proportional to the strength of the emf, which is the amount of work it does per unit charge (measured in V).
- The energy coming in from the emf is balanced by equal energy going out, resulting in a steady state situation. Some of the energy out may be in the form of mechanical work (e.g., a motor) but some (or all) of it is in the form of dissipation through the phenomenon we call *resistance*. The rate of this loss is given by the *Joule heating* formula $P = I^2 R$.
- The detailed energy balance is embodied in Kirchhoff's *loop rule*. As one goes around the circuit tallying the changes in potential, an emf supplying energy is represented by a rise in potential, a resistive element by a fall equal to IR. The total potential change for a complete path is of course zero.
- The other general principle is charge conservation, embodied in Kirchhoff's *junction rule*: the total current coming into any point must equal the total current going out.
- The resistance R of a simple passive element is determined by its geometry and by the microscopic properties of its material, described by the *conductivity*. A thorough account of that quantity had to wait until the development of quantum mechanics.

These are the essential features of the theory of DC circuits.

Induced emf, Lenz's law, and the flux rule.

Courses on this subject usually introduce Faraday's work by discussing the emf produced in a wire moving perpendicular to a B-field. (See the discussion in G.) This is not how Faraday discovered his law, nor does it make clear that anything really new has been discovered. The emf arises from the Lorentz force $q\mathbf{v}\times\mathbf{B}$ on the free electrons in the wire as they are moved through the field.

But when one considers how it looks in the reference frame of the moving wire, one sees that there really is something new. The wire is at rest, the sources of the magnetic field start moving. But suddenly the free charges in the wire begin to move. Why? A force on charges at rest is the signature of an E-field. So somehow the changing magnetic situation is causing an E-field to appear in the wire. That is new.



Shown is the circuit with the moving wire. As is shown in G, the Lorentz force caused by the wire's motion pushes positive charge up the wire, resulting in a counter-clockwise current in the loop containing the resistor. The strength of this "motional" emf is $\mathcal{E} = Bwv$. Now the area of the loop is $A = wx$, and the flux of the B-field into the page is $\Phi = BA = Bwx$. It was noted early on that the rate of change of this flux (because the area changes) is $d\Phi / dt = Bwv$, which is just the emf. So for this case at least we have $\mathcal{E} = d\Phi / dt$.

Is this a more general rule? Suppose the wire is fixed, so the loop area doesn't change, but we increase the strength of the B-field and thus increase the flux. Does this result in an emf? The answer is yes, and the strength is given by the formula we have.

There is one thing still missing from the rule. Which way does the current go? From the Lorentz force argument it is clear: counter-clockwise. But what about a stationary loop but a changing field? We need a rule that ties the direction of the current to the sign of the change in flux. It is given by **Lenz's law**:

> *A current induced by a change in magnetic flux is in such a direction as to oppose the change that caused it.*

In this case, if B increases the flux into the page increases. The induced current, running counter-clockwise, creates its own B-field, which is out of the page inside the loop. This opposes the original increase in flux into the page.

Lenz's law is written into the formula by means of a negative sign. This gives is the **flux rule**:

$$\mathcal{E} = -\frac{d\Phi}{dt}. \qquad (1)$$

Lenz's law is a necessary consequence of conservation of energy. If the induced current went the other way, one would have a runaway situation with energy appearing out of nothing.

Faraday's law.

The flux rule is very useful in dealing with circuits, but it is not a field equation. What is going on at the level of fields is that a changing B-field creates an E-field, one that no longer obeys $\nabla \times \mathbf{E} = 0$. This means that

$$\oint \mathbf{E} \cdot d\mathbf{r} \neq 0.$$

In fact, if the path in the line integral is taken around the loop in the circuit we have been discussing, the integral would give us the work done per unit charge to produce the induced current. This integral is thus the induced emf itself:

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{r}.$$

(If, in addition to the induced E-field, there is another E-field created by charges, the latter will give zero in the integral, so we can use the *total* E-field in the above relation.) Now the change in magnetic flux through an area bounded by the curve is

$$\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{A},$$

so we have

$$\oint \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{A},$$

which is the flux rule written in terms of the fields. If we consider only paths that are fixed in time this becomes

$$\oint \mathbf{E} \cdot d\mathbf{r} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}. \qquad (2)$$

This is the integral form of the field equation we are looking for. Using Stokes's theorem we find the differential form:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \qquad (3)$$

These are the two forms of **Faraday's law**. It is one of the basic equations of the electromagnetic field.

Inductance

This law was not discovered by moving wires in magnetic fields; that came later. The phenomenon that led Faraday (and independently Henry) to discover electromagnetic induction was what we now call *mutual inductance*. Both scientists were experimenting with coils of many turns, creating strong magnetic fields. Faraday noticed that when he turned the current on or off in his coil, a galvanometer (an instrument to measure current) attached to a nearby coil showed deflection, meaning a current was flowing. It deflected one way when the current in his coil was turned on, the other ways when it was turned off. And it was only a brief deflection, disappearing when the current in his coil settled into its final value.

What he had seen was the effect of the flux from his first coil on the second coil, inducing an emf in it if the flux was changing. In terms of the flux rule:

$$\mathcal{E}_2 = -\frac{d}{dt}\Phi_{2/1}.$$

Here $\mathcal{E}_2$ is the emf induced in coil #2 and $\Phi_{2/1}$ is the flux from coil #1 that passes through the area bounded by coil #2. (Actually, if coil #2 has N_2 turns, we should multiply the emf by N_2; it's like batteries in series.)

Of course the flux from coil #1 is proportional to the current that created it, I_1, so $\mathcal{E}_2$ is proportional to dI_1 / dt. The proportionality constant is a property only of the coils and their geometric placement relative to each other. It is called the **mutual inductance** of the system, M_{21}. We have

$$\mathcal{E}_2 = -M_{21}\frac{dI_1}{dt}.$$

One can show (see G) that for the same physical arrangement a change in the current in coil #2 will induce the same emf in coil #1, so $M_{12} = M_{21}$.

Of course the B-field produced by coil #1 also makes flux pass through that coil, so there an induced emf in it too. Thus we also have

$$\mathcal{E}_1 = -L_1\frac{dI_1}{dt},$$

Where L_1 is the **self inductance** of coil #1. In general, then:

$$\mathcal{E}_2 = -M_{21}\frac{dI_1}{dt} - L_2\frac{dI_2}{dt},$$

and similarly for coil #1.

See G for examples. self and mutual inductance play major roles in circuits with time-varying currents, especially those which vary sinusoidally (AC). In our electric power systems, use of mutual inductance in transformers allows power to be transferred over hundreds of miles with relatively little loss to Joule heating.

Magnetic field energy

Consider a single coil which, at $t = 0$, is attached to a battery so a current begins to flow. While the current is rising toward its final value there is an induced emf in the coil:

$$\mathcal{E} = -L\frac{dI}{dt}.$$

(This is often called a ″back emf″ because it opposes the rise of the current.) The battery must do work against this emf in order to get the current going. The power it delivers is

$$P = \mathcal{E} \cdot I = L \cdot I\frac{dI}{dt}.$$

(Because of resistance and other things the total power delivered by the battery may be larger; this part is what is required to set up the magnetic field.) The total work done is the integral of this power over time:

$$W = \tfrac{1}{2}LI^2.$$

This is the energy input to make the B-field, and it can be recovered if the current is switched off, so it is potential energy. We call it the B-field energy, denoted by U_B.

As is shown in G, this energy can be written in terms of the B-field itself, independent of the geometric arrangement that L represents. the formula is

$$U_B = \int d^3r\,\tfrac{1}{2\mu_0}B^2.$$

The integral is (in principle) over all space. The integrand is the *energy density* in the B-field:

$$\eta_b = \tfrac{1}{2\mu_0}B^2.$$

If we have both electric and magnetic fields (but no materials such as dielectrics that alter the fields) then the total electromagnetic energy density is

$$\eta_{em} = \tfrac{1}{2}\varepsilon_0 E^2 + \tfrac{1}{2\mu_0}B^2.$$

The numbers ε_0 and μ_0 are related: $\varepsilon_0\mu_0 = 1/c^2$, where c is the speed of light, so we finally have

$$\eta_{em} = \tfrac{1}{2}\varepsilon_0(E^2 + c^2B^2).$$

This is an important formula about a very important concept: the fields possess energy.

Completion of the theory: displacement current.

As we have it so far, Ampere's law is $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$. This is fine for steady currents, for which $\nabla \cdot \mathbf{j} = 0$, since the divergence of a curl is identically zero. But when the current is not steady (as for example when a capacitor is being charged) then charge conservation, which implies

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0,$$

is incompatible with Ampere's law.

Knowing Gauss's law, Maxwell was able to find the way out of this entirely by theoretical arguments. We have $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$, so $\partial \rho / \partial t = \nabla \cdot (\varepsilon_0 \partial \mathbf{E} / \partial t)$. If we add to the $\mathbf{j}$ on the right side of Ampere's law the term $\varepsilon_0 \partial \mathbf{E} / \partial t$, then we have a quantity whose divergence is always zero. In the process we have also enforced conservation of charge. The result is

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$

The extra term is called the *displacement current* density. It is very small in ordinary situations because $\varepsilon_0 \sim 10^{-11}$ in our units. But of course in empty space where $\mathbf{j} = 0$ it is the only source of a new field.

The complete field equations (without dielectrics or materials with magnetic properties) are the famous Maxwell equations:

$$\varepsilon_0 \nabla \cdot \mathbf{E} = \rho, \; \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \; \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$

To these we add the Lorentz force law for the force on a point charge:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

These five equations embody classical electromagnetism.