Physics 182

Magnetostatics

Overview.

Like electricity, magnetism has ancient roots. Aristotle cites Thales, about 300 years before him, as the first to describe magnetic phenomena scientifically. The first phenomena involved stones found in various places, especially the province of Magnesia, that had the power to attract bits of iron. It was also found that a small bit of iron could be "magnetized" by contact with these stones. Around 1000 CE the ability to determine the direction of north by a magnetized needle was discovered. In 1600 Gilbert explained this by asserting that the earth behaved like a giant magnet itself.

It was not until 1820 that the connection between magnetism and electric currents was discovered, by Oersted. Within a decade the laws governing the creation of magnetic fields by currents, and the action of those fields on currents, had been laid out by Ampere and others. All the laws governing magnetic phenomena of currents that do not change with time had been found. Those laws are the subject of these notes.

The magnetic field.

At the time of Ampere the microscopic picture of electric currents was still vague, because nobody knew how the flow of charge was constituted. By 1900 it was known that a current in a wire is a flow of electrons, and the effect of a magnetic field on a current was understood as a collective effect of the interaction of the field with the moving electrons. This changed the way a magnetic field was defined.

We now define a magnetic field, called **B**, by the force it exerts on a single charged particle. But the charge must be moving for such a force to exist. The definition of **B** comes from the law for that force (often called the Lorentz force), which is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \, .$$

Here q is the charge and **v** its velocity.

There are some interesting features of this force:

- It is perpendicular to both **B** and **v**. The latter means it can do no work, because it is perpendicular to the motion.

- In a uniform B-field a charge moving perpendicular to the field will move at constant speed in a circle of radius $r = mv / qB$. The frequency of revolution in this circle, called the "cyclotron frequency", is $qB/(2\pi m)$, which (for non-relativistic speeds) is independent of v.
- If **v** also has a component along the direction of **B**, that component is unchanged, and the motion will be a helix.

It is a straightforward calculation to find the force that a B-field will exert on an infinitesimal bit of current. This is best expressed in terms of the current *density* **j**, which is a vector field defined by $\mathbf{j} = \rho\mathbf{v}$, where ρ is the charge density and **v** is the average velocity of the motion of the charges. Then the force on an infinitesimal volume of charge is

$$d\mathbf{F} = \rho\mathbf{v}\times\mathbf{B}\,d^3r = \mathbf{j}\times\mathbf{B}\,d^3r\,.$$

If the infinitesimal volume is a bit of wire of area A and length dl then this becomes $d\mathbf{F} = A\mathbf{j}\times\mathbf{B}\,dl$. In ordinary cases the charges move along the wire; we define a vector $d\mathbf{l}$, directed along the wire with magnitude dl, and write $\mathbf{j}\,dl = j\,d\mathbf{l}$. This gives $d\mathbf{F} = jA\,d\mathbf{l}\times\mathbf{B}$. The quantity jA is the rate at which charge crosses area A, and is called the *current I*. This gives the force in a useful form:

$$d\mathbf{F} = I\,d\mathbf{l}\times\mathbf{B}\,.$$

To find the total force exerted by a magnetic field on a set of currents, one integrates the above expression over all the wires in the system.

In formal terms, the current is the flux of **j** through a surface:

$$I = \int \mathbf{j}\cdot d\mathbf{A}\,.$$

If the surface is closed, the current outward through it must equal the decrease in charge within the surface, by conservation of charge, so we have

$$\oint \mathbf{j}\cdot d\mathbf{A} = -\frac{d}{dt}\int \rho\,d^3r\,.$$

But by the divergence theorem

$$\oint \mathbf{j}\cdot d\mathbf{A} = \int \nabla\cdot\mathbf{j}\,d^3r\,,$$

so we find

$$\nabla\cdot\mathbf{j} = -\frac{\partial\rho}{\partial t}\,.$$

This is one of the major laws of electromagnetism, expressing conservation of charge.

Sources of the B-field.

Until 1820 the only magnetic fields known were those from "permanent" magnets, objects mostly made of iron or its compounds. But Oersted's discovery changed all that and made it possible to create magnetic fields for more practical uses. Within months the basic laws governing creation of B-fields by currents had been worked out, especially by Biot and Savart.

The Biot-Savart law is the counterpart to the formula for the E-field of a point charge. It gives the B-field created by an infinitesimal bit of moving charge, as long as the motion is uniform in time. The field thus created is time-independent.

A single moving point charge is *not* uniform in time. What we have in mind is a bit of a steady current in a circuit. The single moving point charge will be treated much later.

The condition of steady current means that $\nabla \cdot \mathbf{j} = 0$. Under that condition the field created by currents is given by

$$\mathbf{B}(\mathbf{r}) = (\text{const.}) \cdot \int d^3 r' \frac{\mathbf{j}(\mathbf{r}') \times \mathbf{R}}{R^3},$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. The constant depends on the units. In SI units the unit of current (and indirectly that of charge) is defined so that the constant is exactly 10^{-7}. It is usually written, however, as $\mu_0 / 4\pi$.

Biot and Savart were dealing with currents in wires. That case is handled in the above equation by making the change $d^3 r' \, \mathbf{j}(\mathbf{r}') \to I(\mathbf{r}') d\mathbf{l}$. We are more interested here in the formal properties of this field than in the fields of specific arrangements of currents.

Field equations for steady currents.

It is an exercise in vector calculus to show that the field given by the above equation satisfies the following:

$$\nabla \cdot \mathbf{B} = 0, \; \nabla \times \mathbf{B} = \mu_0 \mathbf{j}.$$

These are the field equations of the static magnetic field in differential form. The first, a sort of magnetic Gauss's law, tells us that there in no magnetic analog of a single point charge, and that the lines of the field close on themselves. The second says that the sources (the currents) are related to the field by

$$\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int \mathbf{j} \cdot d\mathbf{A} .$$

This field equation is usually called Ampere's law.

Because the field has zero divergence it can be written as the curl of another vector field:

$$\mathbf{B} = \nabla \times \mathbf{A} .$$

This field, **A**, is called the *vector potential*. Like the electrostatic scalar potential. it is not unique. To it can be added the gradient of any scalar, since the curl of a gradient is identically zero. The choice of such a gradient is called choosing a *gauge*, and a change of that choice is a *gauge transformation*.

One popular choice is to make $\nabla \cdot \mathbf{A} = 0$. This choice is called the "Coulomb gauge", because **A** satisfies an equation resembling Poisson's equation:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j} .$$

The solution of this is analogous to the formula for the scalar potential in electrostatics:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{j}(\mathbf{r}')}{R} .$$

Example: a uniform B-field. The field $\mathbf{A}(\mathbf{r}) = \frac{1}{2} \mathbf{B} \times \mathbf{r}$ gives both $\nabla \times \mathbf{A} = \mathbf{B}$ and $\nabla \cdot \mathbf{A} = 0$.

Exercise: Prove these claims.

The integral form of the curl equation, like the integral form of Gauss's law for electricity, can be used to make formal arguments about the field, and, in a few cases of high symmetry, to calculate the field to a good approximation. This equation reads:

$$\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int \mathbf{j} \cdot d\mathbf{A} = \mu_0 I_{linked} .$$

Here I_{linked} is the net current (the flux of **j**) passing through the area bounded by the closed curve over which the line integral is taken. The direction of a positive current is determined by a right hand rule: curl your right hand fingers around the way the closed curve is integrated; your thumb indicates positive current.

There are several examples of this use of Ampere's law worked out in G.

Summary of static fields.

The field equations, up to now, are these:

- Electricity: $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$, $\nabla \times \mathbf{E} = 0$ or $\varepsilon_0 \oint \mathbf{E} \cdot d\mathbf{A} = \int \rho \, d^3r$, $\oint \mathbf{E} \cdot d\mathbf{r} = 0$.

• Magnetism: $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ or $\oint \mathbf{B} \cdot d\mathbf{A} = 0$, $\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int \mathbf{j} \cdot d\mathbf{A}$.

Note that none of the equations involves both **E** and **B**. To these must be added the law specifying the force the fields exert on a single point charge:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

As noted, at this point we require $\nabla \cdot \mathbf{j} = 0$, so only steady currents and time independent charge densities are described by these equations. When those restrictions are relaxed, we find additional terms are needed in the field equations, linking **E** and **B**.