

## Physics 182

# Electrostatics III

### Overview.

We have discussed the electrostatic field in empty space and on and in conductors. But we have not dealt in a systematic way with the field in non-conductors. That is the subject of these notes.

Non-conductors are made of atoms and molecules whose electrons are bound to them, not free to move around within the material. The atoms and molecules themselves have zero total charge, but they are not electrically inert. Atoms alone cannot have an intrinsic electric dipole moment, but molecules can. And in the presence of a field caused by external charges, atoms can distort their charge distributions so as to create an *induced* dipole moment. The fields created by these dipoles are not as strong as fields created by free charges, but they are not negligible.

However, at ordinary temperatures, in the absence of an external E-field the intrinsic dipole moments of molecules become randomly oriented as to direction, so their average effect is zero; and induced moments require an external field. So in their “natural” state non-conductors (or *dielectrics*, as they are usually called) have no overall electric properties.

So our interest is in their properties when there is an externally caused E-field. In simple terms, such a field *polarizes* a dielectric, creating induced dipoles aligned with the field, or — because a dipole moment in an external field experiences a torque tending to align it with the field — aligning intrinsic dipoles in the direction of the field.

We will give the description of this polarization and its consequences. Since solid materials can have structures that are not the same in all directions, the polarization can be somewhat complicated, but we will treat only the simpler cases.

### Induced and aligned dipoles. The polarization field.

If an atom, or a “nonpolar” molecule which has no intrinsic dipole moment, is placed in an E-field, a shift occurs in the average charge distribution. The “clouds” of electrons are moved in the direction opposite to the field and the nuclei is moved in the direction of the field. This means that, seen from outside the atom, the center of positive charge

no longer coincides with the center of negative charge, resulting in a dipole moment parallel to the external field. We write

$$\mathbf{p} = \alpha \mathbf{E}.$$

Here  $\mathbf{E}$  is the field “seen” by the atom, which may not be the same as the field outside the dielectric material, and  $\alpha$  is the *electric polarizability* of the atom. In the simplest cases (to which we will mostly limit our attention)  $\alpha$  is simply a scalar constant.

Actually it is a 2nd rank tensor, and the above equation is  $p_i = \alpha_{ij} E_j$ . In crystalline materials this is often important.

If there are “polar” molecules in the material, the process of alignment of their dipoles is the dominant one. The torque  $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$  gives rise to a potential energy of alignment  $U = -\mathbf{p} \cdot \mathbf{E}$ . The stable equilibrium state is with  $\mathbf{p}$  in the direction of  $\mathbf{E}$ .

In any case, the external field creates or aligns a set of dipole moments. The average dipole moment per unit volume is what is important at the macroscopic level. It is called the *polarization* field:

$$\mathbf{P} = n \langle \mathbf{p} \rangle,$$

where the  $\langle \rangle$  indicates an average over a small volume (that nevertheless contains many atoms or molecules) and  $n$  is the number of atoms per unit volume.

Within a solid dielectric the existence of a non-zero average dipole moment tends to produce no effect because in any given small volume as much positive charge is shifted into the volume as negative charge by the external E-field. But at the surface there are no atoms beyond the surface to compensate for the charge shifts of the atoms at the surface. As a result a net *surface charge* appears on the surfaces. This is the polarization charge, and one can show that it is given by

$$\sigma_{pol} = \mathbf{P} \cdot \mathbf{n}$$

where  $\mathbf{n}$  is a unit vector directed away from the material at the surface.

In a non-uniform E-field, or in a dielectric which does not have the same properties everywhere, then  $\mathbf{P}$  may not be uniform within the material, giving rise to a polarization charge of density

$$\rho_{pol} = -\nabla \cdot \mathbf{P}.$$

The polarization charges create their own E-fields, of course, and thus alter the net field.

Since one does not directly measure the polarization by counting up the dipole moments in a small volume of the material, one describes the effect in terms of the polarization created by a given net E-field. The relation is

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E}_{tot}.$$

Here  $\chi$  is called the *electric susceptibility*.

Here again the general relation is more complicated:  $P_i = \epsilon_0 \chi_{ij} E_j$ . For materials that are *isotropic* (the same in all directions) the tensor is diagonal and we have the simpler relation.

If the material is isotropic, homogeneous (the same at all points in it), and linear ( $\chi$  is a constant independent of  $\mathbf{E}$ ), we call it “simple”. We will deal only with simple materials.

The total field is related, through the polarization, to the externally applied field  $\mathbf{E}_0$ , of course. For simple materials we have  $\mathbf{E}_{tot} = \mathbf{E}_0 - \mathbf{P} / \epsilon_0$ . Using the relation above this gives  $\mathbf{E}_0 = (1 + \chi) \mathbf{E}_{tot}$ . We define the *dielectric constant* by  $\kappa = 1 + \chi$  and have

$$\mathbf{E}_{tot} = \frac{1}{\kappa} \mathbf{E}_0.$$

The dielectric constant is easy to measure, for example by placing the material between the plates of a capacitor and measuring the change in capacitance.

The polarization charge must be included as a source of the E-field, of course. Gauss’s law becomes

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho_{free} + \rho_{pol}.$$

In principle, of course, this and  $\nabla \times \mathbf{E} = 0$  contain all we need to determine the E-field. But because the polarization charges are not usually known in advance, we use other methods.

### Models for the susceptibility.

For atoms or non-polar molecules, the polarization is in the form of induced dipoles, so we need to know the “local” field seen by an individual microscopic particle. This is neither the applied field  $\mathbf{E}_0$  nor the average field. To make a guess as to what it is we imagine a small cavity in the material, within which the particle will reside. The walls of the cavity will contain polarization surface charge, given by  $\mathbf{P} \cdot \mathbf{n}$ , which makes the field in the cavity different from the average field in the material.

The simplest choice of cavity is a small spherical hole. It is not hard to show that the field in such a hole is parallel to the average field (and to  $\mathbf{P}$ ) with magnitude

$$E_{local} = E_{av} + P / 3\epsilon_0.$$

Now  $P$  obeys two relations, one to the polarizability ( $P = n\alpha E_{local}$ ) and one to the susceptibility ( $P = \epsilon_0 \chi E_{av}$ ), so combining these with the equation above we find

$$\chi = \frac{n\alpha / \epsilon_0}{1 - n\alpha / 3\epsilon_0}.$$

This is the Clausius-Mossotti equation, relating the microscopic quantity  $\alpha$  to the macroscopic (measurable) one  $\chi$ .

To use this relation we need a model of the atom or non-polar molecule from which to estimate  $\alpha$ . That means we need quantum mechanics. Simple estimates from, e.g., the Bohr model give  $\alpha \sim \epsilon_0 a^3$ , where  $a$  is the radius of the atom or molecule. This means that  $\alpha / \epsilon_0 \sim 10^{-30} \text{ m}^3$ . In a gas, where  $n \sim 10^{25}$  atoms per  $\text{m}^3$ , we get  $n\alpha / \epsilon_0 \sim 10^{-5}$ . In the C-M equation the denominator is close to 1 and we have  $\chi \sim 10^{-5}$ , which is consistent with experiment. For a solid it is not unusual to have  $n \sim 10^{30}$ , so we must use the C-M equation to make estimates.

### Boundary conditions.

For simple dielectrics we have  $\mathbf{P} = \epsilon_0 \chi \mathbf{E}$ , so Gauss's law reads

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho_{free} + \rho_{pol} = \rho_{free} - \epsilon_0 \chi \nabla \cdot \mathbf{E}.$$

Rearranging and using  $\kappa = 1 + \chi$  we have  $\epsilon_0 \kappa \nabla \cdot \mathbf{E} = \rho_{free}$ . In integral form the two field equations become

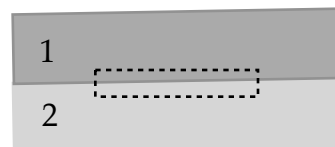
$$\epsilon_0 \oint \kappa \mathbf{E} \cdot d\mathbf{S} = Q_{free}, \quad \oint \mathbf{E} \cdot d\mathbf{r} = 0.$$

Here  $Q_{free}$  is the free charge enclosed in the volume bounded by the closed surface.

What is new about this form of Gauss's law is that the field for which the flux is calculated is not  $\mathbf{E}$  but  $\kappa \mathbf{E}$ . If  $\kappa$  is the same everywhere on the surface, it can be taken outside the integral, and things are simple. But if  $\kappa$  changes for different parts of the surface, we must be careful.

Consider the boundary between two dielectrics (one of which might be the vacuum). We wish to determine how  $\mathbf{E}$  changes as the boundary is crossed.

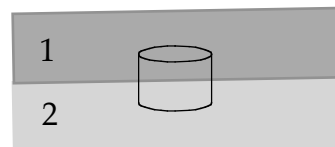
First we use the circulation equation. Shown is the boundary and a closed curve to integrate over. Let the horizontal length of the path be  $\ell$ , and integrate clockwise, starting at the top left corner. Since the path will be shrunk to infinitesimal size, we can assume the field does not change in the horizontal ( $x$ ) direction. We will also shrink the vertical sides to zero, so they will give no contribution. For the top part we get  $E_x(1) \cdot \ell$ ; for the bottom part,  $-E_x(2) \cdot \ell$  (the integration is in the negative  $x$ -direction). The total integral is the sum of these, and it must give zero, so we have, calling the  $x$  component  $E_{\parallel}$  because it is the one parallel to the surface:



$$E_{\parallel}(1) = E_{\parallel}(2).$$

The component of  $\mathbf{E}$  *parallel* to the surface is *continuous* across the boundary.

Now we use Gauss's law, with a pillbox surface as shown. Again we will shrink the vertical dimension to zero, and treat the top and bottom area  $A$  is small enough to assume  $\mathbf{E}$  is constant. For the top we have the contribution  $\kappa_1 E_{\perp}(1) \cdot A$ ,



where  $E_{\perp}$  means the component of  $\mathbf{E}$  perpendicular (upward) to the boundary between the materials. For the bottom we have  $-\kappa_2 E_{\perp}(2) \cdot A$ . If there is no free charge in the situation, then the flux integral vanishes, so we find

$$\kappa_1 E_{\perp}(1) = \kappa_2 E_{\perp}(2).$$

The component of  $\kappa \mathbf{E}$  *perpendicular* to the surface is *continuous* across the boundary.

These two statements are what we need in many situations involving dielectrics with no free charge.

One is often solving Laplace's equation for the potential and must apply the boundary conditions at boundaries between dielectrics. The conditions on  $\mathbf{E}$  become conditions on  $V$ :

- (1) The potential  $V$  is continuous across the boundary.
- (2) The quantity  $\kappa \frac{\partial V}{\partial r_n}$  is continuous across the boundary, where  $r_n$  is the coordinate perpendicular to the boundary.

There are several examples of the use of these conditions worked out in G.