Physics 182

Electrostatics II

Overview.

In the late 18th and early 19th centuries remarkable progress was made in the study of differential equations, especially those that applied directly to problems in physics and engineering. Some of the most famous names in the history of mathematics are associated with these efforts: Lagrange, Laplace, Poisson, to name only those who worked mainly in France. The "paradigm shift" responsible for this new progress was the discovery that energy is more important than force in describing physical systems, and that the proper way to think about interactions is in terms of fields.

In the case of gravity and electrostatics (which are mathematically very similar) attention was focused on properties of the potentials., scalar fields obeying linear partial differential equations. In the case of electrostatics, our main interest, the equations are those of Poisson and Laplace. These arise out of Gauss's law for the E-field:

$$\nabla \cdot \mathbf{E} = 4\pi k \rho \,.$$

We use the electrostatic connection between $\mathbf{E}$ and V: $\mathbf{E} = -\nabla V$. This gives Poisson's equation for V:

$$\nabla \cdot \nabla V = -4\pi k \rho \,.$$

The derivatives on the left side are $\partial_i \partial_i V$, which is conventionally written as ∇^2. In cartesian coordinates Poisson's equation is

$$(\partial_x^2 + \partial_y^2 + \partial_z^2) V = -4\pi k \rho \,.$$

In other coordinate systems (e.g., spherical) the left side is more complicated.

Poisson's equation is a linear second order partial differential equation. It is inhomogeneous (the term on the right side does not contain V of its derivatives). This means that the general solution is a particular solution plus any solution of the homogeneous equation with zero on the right side. Thus our focus turns to that equation, which is Laplace's equation:

$$\nabla^2 V = 0 \,.$$

The study of the solutions of this equation, subject to various boundary conditions, is called *potential theory*. We will discuss some of the major findings of this study.

We will also discuss an expansion of the potential as a series in inverse powers of the distance r between the sources (assumed to occupy a finite region of space) and the field point. This is the *multipole expansion*.

Laplace's equation in cartesian coordinates.

The standard method of finding solutions to Laplace's equation is called *separation of variables*. One looks for a solution that is the product of functions of only one variable. In (x,y,z) coordinates, this means trying a solution of the form $V(x,y,z) = f(x)g(y)h(z)$.

Substitution gives

$$gh \cdot f''(x) + fh \cdot g''(y) + fg \cdot h''(z) = 0\,,$$

where the double-primes mean second derivatives with respect to the relevant variables. Dividing by fgh we have

$$\frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h} = 0\,.$$

For this to hold for all values of all three variables, we must have each of these terms equal to a constant, and the sum of the constants equal to zero:

$$\begin{aligned} f'' &= \alpha f \\ g'' &= \beta g \\ h'' &= \gamma h \\ \alpha + \beta + \gamma &= 0 \end{aligned}$$

Clearly at least one of the constants must be positive and at least one negative.

Now an equation like $f'' = \alpha f$ has well known solutions. If $\alpha > 0$ the solutions are exponentials: $f(x) = Ae^{\pm\sqrt{\alpha}\cdot x}$, where A is some constant. If $\alpha < 0$ the solutions are sines and cosines, or $f(x) = Ae^{\pm i\sqrt{|\alpha|}\cdot x}$. The former are rising or falling functions, the latter oscillate. So we see that the general solution rises or falls in at least one coordinate, and oscillates in at least one.

Of course there is an infinite number of sets of "separation constants" (α, β, γ) that satisfy $\alpha + \beta + \gamma = 0$. But the boundary conditions reduce the number of possibilities, perhaps to only one or a few. We will look at a case in the examples.

Spherical coordinates.

For good diagrams and a general discussion of these coordinates, see G, Sec. 1.4. Their relation to cartesian coordinates are as follows:

$$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi,\ \ z = r\cos\theta.$$

To work out what the differential operator ∇^2 is in these coordinates is not simple, nor is the answer:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

We try to find solutions to $\nabla^2\Psi = 0$ of the form $\Psi(r,\theta,\phi) = f(r)\cdot g(\theta)\cdot h(\phi)$. Substituting and dividing by fgh we manage to separate the part dependent on r:

$$\frac{1}{f}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{g\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right) + \frac{1}{h\sin^2\theta}\frac{\partial^2 h}{\partial\phi^2} = 0.$$

This gives, calling the separation constant α:

$$\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = \alpha f, \qquad (1)$$

$$\frac{\sin\theta}{g}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right) + \alpha\sin^2\theta + \frac{1}{h}\frac{\partial^2 h}{\partial\phi^2} = 0.$$

Not the second equation can be separated, with separation constant β:

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right) + \alpha\sin^2\theta\cdot g + \beta g = 0, \qquad (2)$$

$$\frac{d^2 h}{d\phi^2} = \beta h. \qquad (3)$$

Eqs (1-3) are ordinary differential equations. Two of the variables are angles, with $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. Of course $\phi = 0$ and $\phi = 2\pi$ are the same point in space, so we must have $h(2\pi) = h(0)$ for single-valuedness. This can only happen if h is an oscillating function rather than a real exponential, which in turn requires $\beta < 0$. We write $\beta = -m^2$ and find the solutions $h(\phi) \sim e^{\pm im\phi}$. The single-valuedness requirement then means that m must be an integer (positive, negative, or zero).

That takes care of Eq (3). In Eq (2) we change the variable: $\mu = \cos\theta$. Then after some effort we find

$$(1-\mu^2)\frac{d^2g}{d\mu^2} - 2\mu\frac{dg}{d\mu} + \left[\alpha - \frac{m^2}{1-\mu^2}\right]g = 0.$$

Of course $-1 \le \mu \le 1$. This is a famous equation, named for Legendre. He found that most of its solutions are singular (go to ∞) at $\mu = \pm 1$. The ones that are well-behaved require that $\alpha = \ell(\ell+1)$ where $\ell = 0,1,2,...$ and also require that $|m| \le \ell$. Further, he found that these solutions are polynomials of order ℓ in μ. They are called the *associated Legendre polynomials*, $P_\ell^m(\mu)$. In the special case where $m = 0$ they become simply the *Legendre polynomials* $P_\ell(\mu) = P_\ell^0(\mu)$

Returning to Eq (1) with the new value of α, we have

$$\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = \ell(\ell+1)f.$$

The two solutions of this are r^ℓ and $r^{-(\ell+1)}$, as is easy to show.

Putting it all together, for given ℓ and m we have the solution

$$\Psi_{\ell,m}(r,\theta,\phi) = [A_{\ell,m}r^\ell + B_{\ell,m}r^{-(\ell+1)}]P_\ell^m(\cos\theta)e^{im\phi}.$$

Here $A_{\ell,m}$ and $B_{\ell,m}$ are constants to be fixed by boundary conditions. The most general well-behaved solution is a sum of these functions over all allowed values of ℓ and m.

In situations with axial symmetry, we can choose the *z*-axis to be the symmetry axis. Then the solutions must be independent of ϕ, so only $m = 0$ solutions occur. Most of our cases will be like that. Of course if there is spherical symmetry, the solutions cannot depend on θ either, and only $\ell = 0$ is allowed ($P_0(\mu) = 1$). In that case $\Psi(r) = A + B \cdot \frac{1}{r}$.

There is a more complete discussion in G, with a list of a some Legendre polynomials.

The multipole expansion

In realistic situations the charges that are the sources of a static E-fled occupy a limited region of space. At large distance from that region the field is dominated by a few of the simpler aspects of the charge distribution. The simples is its total charge: if that is not zero, then at a great distance the field is approximately that of a distant point charge.

The finer details of the field so exist, of course, but they can be displayed in a series of terms that become smaller as the distance becomes larger. That series is called the *multipole expansion*.

We start from the general formula for the potential:

$$V(\mathbf{r}) = k\int d^3r'\,\frac{\rho(\mathbf{r}')}{R},$$

where as usual $R = |\mathbf{r}-\mathbf{r}'| = \sqrt{(r_i - r_i')(r_i - r_i')}$ (sum over i of course). We make a Taylor expansion about $\mathbf{r}' = 0$:

$$\frac{1}{R} = \frac{1}{r} + r_i'\left(\frac{\partial}{\partial r_i'}\frac{1}{R}\right)_0 + \tfrac{1}{2}r_i'r_j'\left(\frac{\partial}{\partial r_i'}\frac{\partial}{\partial r_j'}\frac{1}{R}\right)_0 + \ldots$$

The objects $(\ldots)_0$ depend only on $\mathbf{r}$, the field point. The first two, for example, are

$$\left(\frac{\partial}{\partial r_i'}\frac{1}{R}\right)_0 = \frac{r_i}{r^3},$$

$$\left(\frac{\partial}{\partial r_i'}\frac{\partial}{\partial r_j'}\frac{1}{R}\right)_0 = \frac{3r_ir_j - r^2\delta_{ij}}{r^5}.$$

Exercise: Derive these.

Substituting in the formula for V we have

$$V(\mathbf{r}) = k\frac{1}{r}\cdot\int d^3r'\,\rho(r') + k\frac{r_i}{r^3}\cdot\int d^3r'\,\rho(r')r_i' + k\frac{3r_ir_j - r^2\delta_{ij}}{2r^5}\cdot\int d^3r'\,\rho(r')r_i'r_j' + \ldots$$

This is the series we want. The integrals are just numerical quantities for a given charge distribution. The first one is that total charge, for example:

$$Q_{tot} = \int d^3r'\,\rho(r').$$

The second integral is the electric *dipole moment*, a vector:

$$p_i = \int d^3r'\,\rho(r')r_i'.$$

The third integral, a second rank tensor, is the electric *quadrupole moment*:

$$Q_{ij} = \int d^3r'\,\rho(r')r_i'r_j'.$$

Using these, the first three terms of the series give

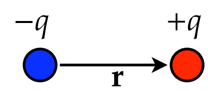
$$V(\mathbf{r}) = k\frac{Q_{tot}}{r} + k\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + k\frac{Q_{ij}(3r_ir_j - r^2\delta_{ij})}{2r^5}\ldots$$

For large *r*, the first term (proportional to r^{-1}) is dominant unless $Q_{tot} = 0$, in which case the second term (proportional to r^{-2}) is dominant unless , and so on.

From $\mathbf{E} = -\nabla V$ one can work out the corresponding terms in the series for the E-field, but the become messy to look at very quickly.

In the case of a neutral object like an atom or molecule, the dominant term from outside is the dipole. In our analysis of dielectrics, it is the fields of these dipoles that will dominate the represent the electric effects of the material.

The simplest charge distribution that has zero charge but a non-zero dipole moment is a pair of equal and opposite point charges separated by a distance, as shown. The dipole moment is $\mathbf{p} = q\mathbf{r}$.



One can similarly make out of point charges a distribution that has no charge or dipole moment but has a quadrupole moment.