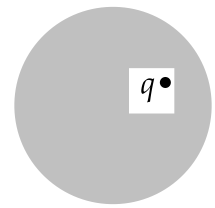
Physics 182

Midterm Exam Solutions

1. A spherical conductor has a cubic cavity in it as shown, within which is a point charge q. How much total charge is on the walls of the cavity? Give a proof of your answer, using Gauss's law.



Consider a Gaussian surface surrounding the cavity but still inside the conductor. Because $\mathbf{E} = 0$ in the conductor, the flux is zero so the enclosed charge is zero. Thus the total charge on the walls of the cavity is $-q$.

2. If the total charge of the conducting sphere is zero, there is charge on the outer surface equal to the negative of your answer to the previous question. How is this charge distributed over the surface?

This charge is distributed uniformly over the surface, because of the spherical symmetry of the surface.

3. Which of the field equations we have so far forbids E-field lines to close on themselves? Give an argument based on that equation to show this.

The equation is $\oint \mathbf{E} \cdot d\mathbf{r} = 0$. If field lines formed closed curves, one could choose a field line for the curve integrated over. But then $\mathbf{E} \cdot d\mathbf{r} = E\,dr$ at all points on the curve, and this is always positive, to the integral could not be zero.

4. Which of the field equations we have so far denies the existence of the magnetic equivalent of a single point charge? Give an argument based on that equation to show this.

The equation is $\oint \mathbf{B} \cdot d\mathbf{A} = 0$. If there were a point source of the B-field, we could choose a surface surrounding only that source. The flux would not be zero.

5. The potential energy of a small electric dipole of constant moment **p** in an E-field due to other charges is $U = -\mathbf{p} \cdot \mathbf{E}$.

   a. Use the identity $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$ to show that the force on the dipole is given by $\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}$.

   b. Show that if $\mathbf{p} \parallel \mathbf{E}$ the force is in the direction of increasing *E*.

   c. Show that if $-\mathbf{p} \parallel \mathbf{E}$ the force is in the direction of decreasing *E*.

   d. Use these results to explain how a charged objects attracts neutral objects, e.g., a charged rod attracts uncharged scraps of paper or metal foil.

a. The force is $\mathbf{F} = -\nabla U = \nabla(\mathbf{p} \cdot \mathbf{E})$. Using the identity and the facts that **p** is constant and $\nabla \times \mathbf{E} = 0$, we have the result claimed.

b. Let **E** be in the *x*-direction. Then we have $F_x = p\, dE / dx$. If dE / dx is positive, so is F_x, and the force is toward increasing *E*. If dE / dx is negative, so is F_x, and again the force is toward increasing *E*.

c. In this case $F_x = -p\, dE / dx$, so the cases are reversed.

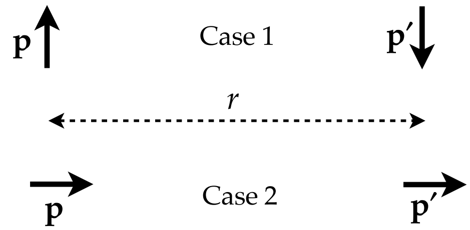
d. The field of the rod either induces or aligns dipole moments in the scraps along the direction of **E**, creating the situation in (b).

6. Consider the interaction of two identical small electric dipoles at a distance large compared to the size of the dipole. Let the dipole moments have magnitude *p*.

   a. Start from the potential of a dipole, given by $V_{dipole}(\mathbf{r}) = k\dfrac{\mathbf{p} \cdot \mathbf{r}}{r^3}$, where **r** is the location of the field point relative to the location of the dipole. Show that the E-field of this dipole is given by $\mathbf{E}(\mathbf{r}) = \dfrac{k}{r^3}\left(\dfrac{3(\mathbf{p} \cdot \mathbf{r})\mathbf{r}}{r^2} - \mathbf{p}\right)$. [Write things in terms of the components of the vectors. For example, $E_i = -\partial_i V$.]

We have $E_i = -k\partial_i[p_j r_j / r^3] = -kp_j \partial_i(r_j / r^3)$. But $\partial_i(r_j / r^3) = \delta_{ij} / r^3 - 3r_j r_i / r^5$. Putting the pieces together we get the result claimed.

b. The potential energy of the second dipole in the field of the first is given by $U = -\mathbf{p}' \cdot \mathbf{E}$. Consider the two configurations shown. In Case 1 $\mathbf{r}$ is perpendicular to both dipoles; in Case 2 it is parallel to both. Find the potential energy in each case.

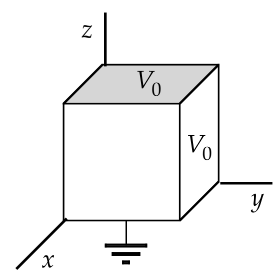


In Case 1 we have $\mathbf{p}' = -\mathbf{p}$ and $\mathbf{r} \cdot \mathbf{p} = 0$, so $U = -kp^2 / r^3$. In Case 2 we have $\mathbf{p}' = \mathbf{p}$ and $\mathbf{r} \cdot \mathbf{p} = rp$, so $U = -2kp^2 / r^3$.

c. In which case is the attractive force between the dipoles stronger? Explain.

Clearly Case 2 gives a stronger attractive force, by a factor of 2.

7. Shown is a conducting cube similar to the one in Assignment 2. But now both the top and the right side are at potential V_0, while the rest of the cube is grounded. One would again use separation of variables to find the potential, looking for solutions of the form $V(x, y, z) = X(x) \cdot Y(y) \cdot Z(z)$.



a. Carry out the substitution into Laplace's equation and get the separated equations in terms of the separation constants.

Substituting into Laplace's equation we have $X'' \cdot YZ + Y'' \cdot XZ + Z'' \cdot XY = 0$. Dividing by XYZ we have $(X''/X) + (Y''/Y) + (Z''/Z) = 0$. Since each quantity in ( ) depends only on one variable, each must equal a constant and the cum of the constants must be zero. So we have $X'' = \alpha X$, $Y'' = \beta Y$, $Z'' = \gamma Z$, and $\alpha + \beta + \gamma = 0$.

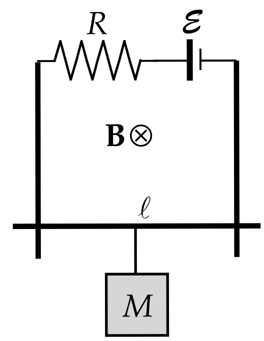
b, c. Which of the functions X, Y, Z will be oscillatory and which exponential? Which separation constants will be positive and which negative? Explain your choices.

We have $\alpha < 0$, $\beta > 0$ and $\gamma > 0$. X must oscillate because it is zero at both $x = 0$ and $x = \pi$. Y and Z must not oscillate because they change from zero to V_0.

8. A primitive motor consists of a horizontal conducting bar that slides on *vertical* conducting rails as shown. The bar is in a uniform magnetic field, so there is an upward force on it, which can lift the block shown.

   a. What is the minimum emf $\mathcal{E}$ that will result in the block being held at rest?

   b. If the block is moving upward at constant speed the net force on it is still zero, so the magnetic force on the bar is the same as in (a). But the block's gravitational potential energy is increasing. Does this increase in energy come from work done by the magnetic forces on the charges constituting the current? Explain.



a. We must have the magnetic force $I\ell B = mg$, where $I = \mathcal{E} / R$, so $\mathcal{E} = mgR / \ell B$.

b. No. Magnetic fields never do work on moving charges. What happens is that the magnetic force on the (positive) charges gives their velocities an upward component. This would reduce the current, which involves only the horizontal component. So the battery must provide more emf to keep the current constant. It is that energy that raises the block. [The effect is described by the back emf arising from Faraday's law on account of the motion of the bar.]