Final Exam

Solutions

1. All of the following statements about lines of the electrostatic field are true, but which one follows from the field equation $\oint \mathbf{E} \cdot d\mathbf{A} = 4\pi k Q_{enc}$? Explain why.

   a. Field lines do not cross each other.

   b. Field lines start on positive charges and end on negative charges.

   c. Field lines do not form closed curves.

   d. Field lines do not penetrate a conductor.

The answer is (b). Positive flux comes from field lines going out through the surface, which are associated with positive charge inside. Negative flux comes from field lines going into the surface, associated with negative charge inside.

2. Concerning the relationship between **E** and V, which of these is wrong? Explain.

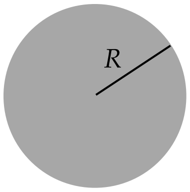
   a. If V is constant at all points of a solid object, then $\mathbf{E} = 0$ within the object.

   b. If **E** is not zero at a point, then V cannot be zero at that point.

   c. As one moves away from a positive charge both V and the magnitude of **E** decrease.

   d. If V depends only on the coordinate variables x and y then $E_z = 0$.

The answer is (b). The potential's value at any point depends on the choice of where it is equal to zero. One could choose the point in question to be that point.



3. A non-conducting sphere of radius R is uniformly charged. The potential at its surface (taking potential to be zero at infinite distance) is V_0. Give all answers in terms of R and V_0.

   a. What is the total charge of the sphere?

   b. What is the field $E(r)$ inside the sphere, i.e., for $r < R$?

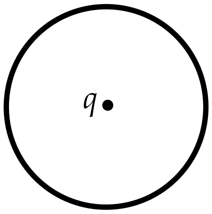
   c. What is the potential at the center?

a. The fields outside are those of a point charge, so $V_0 = kQ/R$, or $Q = RV_0/k$.

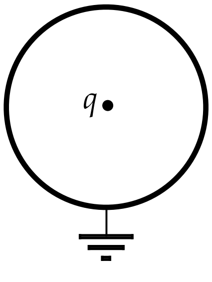
b. By Gauss's law $E(r) = kQ(r)/r^2$, where $Q(r)$ is the charge in a sphere of radius r. In this case $Q(r) = Q \cdot (r/R)^3$, so $E(r) = kQ \cdot r/R^3 = V_0 \cdot r/R^2$.

c. We have $V(0) - V_0 = -\int_R^0 E \cdot dr = -(V_0/R^2)\int_R^0 r \cdot dr = \frac{1}{2}V_0$, so $V(0) = \frac{3}{2}V_0$.

4. "Grounding" means connecting a conductor to the earth, a very large conductor which can accept or supply significant amounts of charge without changing its potential — which one usually chooses to be zero. "Shielding" means surrounding a region of space by a grounded conductor. Some examples:



   a. An uncharged conducting spherical shell surrounds a point charge as q shown. Charge q appears on the outer surface of the shell. (Why?) What are the E-field and potential *outside* the shell? Does it matter whether the point charge is at the center of the shell?



   b. Now we ground the shell. (The symbol is the circuit diagram icon for grounding.) The E-field outside the shell is now zero. What is the charge on the outer surface?

   c. In this situation the interior of the shell is shielded. Suppose we bring up a second point charge Q and place it near the shell but outside it. How does the grounded shell manage to remain at potential zero?

   d. There is a force between the shell system and the charge Q. Is it an attraction or a repulsion? Does it depend in any way on q (its magnitude, its sign, or its location within the shell)?

a. The total charge on the inner surface of the shell is $-q$, so that on the outer surface is $+q$. The fields outside are those of a point charge $+q$ at the center, so $E(r)=kq/r^2$ and $V(r)=kq/r$. It does not matter where the point charge in the cavity is located.

b. The charge on the outer surface runs off to be spread over the earth, so there is no charge on that surface now.

c. Some negative charge is brought up from the earth and spreads over the part of the outer surface that is close to the new charge.

d. The force is an attraction (opposite charges). It does not depend on what is in the cavity.

5. Questions about energy in the E-field.

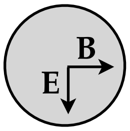
   a. A balloon carries charge Q spread uniformly over its outer surface. If its radius is increased by blowing it up, what happens to the total stored energy in the E-field? [Does it increase or decrease?]

   b. Two spheres, each of radius R, carry charge Q. Sphere A is a conductor, while sphere B is a non-conductor and the charge is distributed uniformly through its volume. For which one is the energy stored in the E-field greater? Explain.

a. The energy decreases. At points outside the balloon the field is unchanged from what they were, but the points that are now inside have zero E-field. The field energy associated with those points has disappeared.

b. The energy is greater for the non-conductor. The field outside the sphere is the same for both, but inside the conductor the field is zero, while it is non-zero for the non-conductor (except at the center).

6. An electromagnetic ion pump has crossed E and B fields as shown in cross-section. The B-field is static, but the E-field can be turned on and off at regular intervals. The ions in the fluid move in response to these fields. Viscosity dampens the flow.



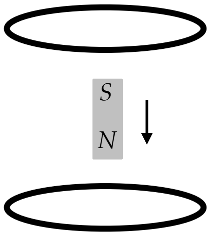
   a. Show that soon after the E-field is turned on a positive ion will experience a force with a component out of the page.

   b. What happens if the ion is negative?

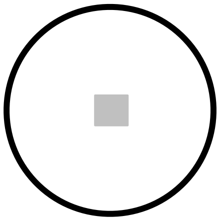
a. The E-field with cause the positive ion to move downward. Its motion in the B-field will cause a force on it ($\mathbf{F} = q\mathbf{v}\times\mathbf{B}$) that is out of the page.

b. A negative ion will move upward, but the magnetic force will still be out of the page.

7. A small bar magnet falls downward between two fixed loops of wire. The field lines of the magnet leave it at the bottom and enter it at the top.



Perspective view



View from above

   a. In the view from above, what is the direction of the current induced in the top loop? [Clockwise or counter-clockwise.] Explain how you know.

   b. Answer the same question for the bottom loop.

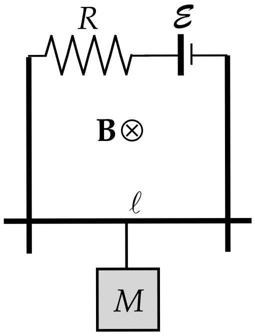
   c. What direction (up or down in perspective view) is the force exerted on the magnet by the loops? Explain how you know. [The loops have magnetic moments which act like bar magnets.]

a. As viewed from above, the bottom loop has increasing magnetic flux into the page. This induces a current which by Lenz's law, is clockwise.

b. The top loop has decreasing flux into the page, so its induced current is counter-clockwise.

c. The magnetic moment of the bottom loop is toward the magnet, so the force between them is a repulsion. (Like magnetic poles facing each other.) The magnetic moment of the top loop is also toward the magnet, so the force is an attraction. (Unlike poles facing each other.)

8. Shown is a primitive motor. The B-field is uniform.



   a. Find the emf $\mathcal{E}_0$ required to hold the block at rest.

   b. If the block moves upward at speed v, what is the induced "back emf" $\mathcal{E}'$ in the circuit?

   c. What must be the emf supplied by the battery to keep the block moving at constant speed?

   d. The additional emf implies an additional amount of power $\mathcal{E}'I$ delivered by the battery. Show that it is exactly the power that lifts the block. [Recall that power input from a force is $P = \mathbf{F}\cdot\mathbf{v}$.]

a. The magnetic force must equal mg so we have $I\ell B = mg$, or $I = mg / B\ell$. Since $\mathcal{E}_0 = IR$ we find $\mathcal{E}_0 = mgR / B\ell$.

b. The moving bar produces an induced emf $\mathcal{E}' = B\ell v$, opposite to that of the battery.

c. The battery must supply $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}' = \mathcal{E}_0 + B\ell v$ to keep the current at its value in (a) so that the total force on the block remains zero.

d. We have $\mathcal{E}'I = B\ell v \cdot mg / B\ell = mgv$. This is the power required to lift the block.

9. A solenoid has N_1 total turns, cross-section area A, length ℓ, and negligible resistance. The field inside its windings is (approximately) $B(t) = \mu_0 (N_1 / \ell) I(t)$, where $I(t) = I_0 \cos\omega t$.

   a. Find the total induced emf $\mathcal{E}_1(t)$ in the solenoid assuming the B-field is uniform over its whole length. [What is the induced emf in each turn?]

   b. Now a shorter "secondary" solenoid of slightly greater cross-section is wrapped around the "primary" one near its center. This secondary solenoid has N_2 turns. Find the total induced emf $\mathcal{E}_2(t)$ in this solenoid.

   c. Find the ratio $\mathcal{E}_2(t) / \mathcal{E}_1(t)$. (This is the law of the transformer).

a. For each turn the flux is $\Phi_1(t) = A \cdot B(t) = (\mu_0 A N_1 / \ell) \cdot I(t)$. We have $d\Phi_1 / dt = (\mu_0 A N_1 / \ell) \cdot dI / dt$, so the total emf is $\mathcal{E}_1 = -N_1 d\Phi_1 / dt$, or $\mathcal{E}_1 = -(\mu_0 A N_1^2 / \ell) \cdot dI / dt = (\mu_0 A N_1^2 / \ell) \cdot \omega \sin\omega t$.

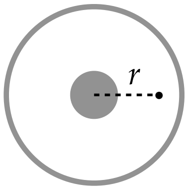
b. The area of the new coil is the same, so the flux through each turn is the same. The total emf is $\mathcal{E}_2 = -N_2 d\Phi_1 / dt = -(\mu_0 A N_1 N_2) \cdot dI / dt = (\mu_0 A N_1 N_2) \cdot \omega \sin \omega t$.

c. The ratio is $\mathcal{E}_2 / \mathcal{E}_1 = N_2 / N_1$.

10. A coaxial cable connects a battery and a resistor. The higher potential is on the inner conductor. Also shown is a cross-section view of the cable. You are to analyze the power given to the resistor in terms of its flow through the cable.

a. At the point shown in cross-section, indicate the directions of **E**, **B** and **S**.

b. Use Gauss's law to find the E-field at the point shown between the conductors, in terms of $\mathcal{E}$. The inner conductor has radius *a* and the outer one radius *b*.



c. Use Ampere's law to find the B-field at the same point, in terms of $\mathcal{E}$ and *R*.

d. What is the Poynting vector **S** (magnitude and direction) at that point?

e. Find the total flux of **S** across the area of the gap between the conductors, to show that the total power carried by the fields is $\mathcal{E}^2 / R$.

a. The potential of the inner conductor is higher, so **E** is directed radially out from the axis at the point. The current is into the page, so **B** is clockwise, therefore down at the point. This makes **S** into the page.

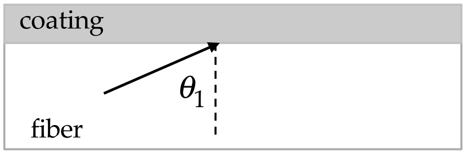
b. Use a cylinder with radius *r* and length ℓ. The flux through the surface is $2\pi r \ell \cdot E$. The charge enclosed is $\lambda \ell$, where λ is the charge per unit length. Thus $E(r) = 2k\lambda / r$. To put this in terms of the potential difference $\mathcal{E}$ we integrate: $\mathcal{E} = -\int_b^a E \cdot dr = 2k\lambda \cdot \ln(b/a)$. This gives $E(r) = \dfrac{\mathcal{E}}{\ln(b/a)} \dfrac{1}{r}$.

c. Use a circle of radius *r*. Then $2\pi r \cdot B = \mu_0 I$, or $B(r) = \mu_0 I / 2\pi r$.

d. We have $S = \dfrac{EB}{\mu_0} = \dfrac{\mathcal{E} I}{\ln(b/a)} \dfrac{1}{2\pi r^2}$. Direction given in (a).

e. Use a ring of radius r and width dr. Then $\int \mathbf{S}\cdot d\mathbf{A} = \int S\cdot 2\pi r\cdot dr = \frac{\mathcal{E}I}{\ln(b/a)}\int_a^b \frac{1}{r}dr = \mathcal{E}I$. Since $I = \mathcal{E}/R$, this gives the result.

11. When an e-m wave is incident on the interface between two transparent media, the law of refraction gives the direction of the wave transmitted into the new medium: $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where θ_1 and θ_2 are the angles (measured form the normal to the interface) giving the directions of the incident and reflected waves, and $n = c/v$ is the index of refraction of the corresponding medium. If $n_2 < n_1$ there are angles of incidence θ_1 such that the law of refraction would require $\sin\theta_2 > 1$, which is impossible. For these angles of incidence there is no refracted wave, and the light is totally reflected. The smallest such angle is called the critical angle, defined by $\sin\theta_c = n_2/n_1$. The phenomenon of total reflection is the basis of fiber optics.



Optical fibers are often coated with a transparent material of a different index of refraction. Shown is the surface of such a fiber. The fiber has refractive index n and the coating has index n'. Above the coating is air ($n = 1$). We wish to show that the coating has no net effect on total reflection.

a. Let θ_1 in the case shown be the critical angle for a fiber-air interface. Find the angle θ_2 at which the refracted ray will strike the coating-air surface.

b. Show that θ_2 is the critical angle for the coating-air interface, so there will be total reflection at that surface.

c. Show that if θ_1 is made larger, the light is either totally reflected at the fiber-coating interface or at the coating-air interface.

a. The law of refraction gives $\sin\theta_2 = (n/n')\cdot\sin\theta_1$.

b. Since $\sin\theta_1 = 1/n$ we have $\sin\theta_2 = 1/n'$, so θ_2 is the critical angle.

c. If θ_1 is increased so is θ_2, and total reflection still occurs. If $n' < n$ there will be values of θ_1 for which there is total reflection at the fiber-coating interface.

12. Lightning strikes involve ionizing the air to create a conducting path for the exchange of charge between the cloud and the ground, which typically have a potential difference of about 10^8 V. To ionize the air requires an E-field of about 10^6 V/m. The charge exchange involves a current of about 10^4 A, lasting only about 10^{-4} s. Here are two questions to be answered in terms of the principles of electromagnetism.

    a. It is well known that lightning tends to strike the tallest objects. Why is this so? [Things in contact with the ground are all at more or less the same potential.]

    b. It is not unusual for a house near a tree which is struck by lightning to experience severe damage to its electrical wiring and things connected to it, occasionally even catching fire, while the house itself was not struck. What causes this?

a. The E-field is roughly $\Delta V / d$, where ΔV is the potential difference between the cloud and the ground, and d is the distance to the cloud. So the field is largest when d is smallest, which is why lightning tends to strike tall objects.

b. There is a large and rapidly changing B-field due to the current. This induces emfs in the household wiring loop that can cause rapid heating and even fires.