Physics 182

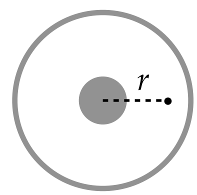
Assignment 5

1. The coaxial cable shown connects a battery and a resistor. The higher potential is on the inner conductor. Also shown is a cross-section view of the cable. You are to analyze the power given by the battery to the resistor in terms of its flow through the cable.

a. At the point shown in cross-section, indicate the directions of **E**, **B** and **S**.

E is to the right (radially away form the inner conductor), **B** is down (the lines are clockwise), so **S** is into the page.

b. Use Gauss's law to find the E-field at the point shown between the conductors, in terms of $\mathcal{E}$. The inner conductor has radius a and the outer one radius b. *Ans*: $E = \dfrac{\mathcal{E}}{\ln(b/a)} \cdot \dfrac{1}{r}$.



Use a cylindrical surface of radius r and length ℓ. By symmetry the flux is $E \cdot 2\pi r \cdot \ell$. The enclosed charge is $\lambda\ell$, where λ is the charge per unit length. This leads to $E = \dfrac{\lambda}{2\pi\varepsilon_0}\dfrac{1}{r}$. To relate it to the potential difference $\mathcal{E}$ integrate: $\mathcal{E} = \int_a^b E\,dr = \dfrac{\lambda}{2\pi\varepsilon_0}\ln(b/a)$. This leads to the result claimed.

c. Use Ampere's law to find the B-field at the same point. Ans: $B = \dfrac{\mu_0 I}{2\pi r}$.

This has the same symmetry as a long wire, and the answer is the same

d. Find the magnitude of **S** at the point.

We find $S = \dfrac{\mathcal{E} I}{2\pi \cdot \ln(b/a)} \dfrac{1}{r^2}$

e. Integrate the flux of **S** to find the power passing down the cable. Use as element of area a circular ring of radius r and width dr. *Ans*: $P = \mathcal{E} I$.

The flux is $\int \mathbf{S} \cdot d\mathbf{A} = \dfrac{\mathcal{E} I}{2\pi \cdot \ln(b/a)} \int_a^b \dfrac{2\pi r\, dr}{r^2} = \mathcal{E} I$ as claimed.

2. A long thin solenoid has a circular cross-section of radius a, and is wound with n turns per unit length. Its turns have current I_0, which is varying with time.

Around it coaxially is a circular loop of wire of radius $b >> a$, and resistance R.

a. Let I_0 be changing with time at rate dI_0 / dt. Find the current induced in the loop.

By Faraday's law the induced E-field at points outside the solenoid is given by $2\pi r \cdot E(r) = \pi a^2 \cdot \partial B / \partial t$, so we have $E(r) = A / r$, where $A = \frac{1}{2} a^2 \cdot \partial B / \partial t$. The emf in the loop is $\mathcal{E} = 2\pi b \cdot E(b) = 2\pi A$. The current is thus $I = \mathcal{E} / R = 2\pi A / R$.

b. Consider a point on the outside surface of the solenoid, at a distance x from the plane of the loop. Find the induced field **E**.

Here the field is $E(a) = A / a$, independent of x.

c. What is the field **B** at that point due to the ring's current? [Use the field along the axis of a loop. See G, Chap 5.]

Along ghe axis of he loop ad distance x from the center the field is

$$B(x) = \frac{\mu_0 I}{2} \frac{b^2}{(b^2 + x^2)^{3/2}} = \frac{\mu_0 \pi A}{R} \frac{b^2}{(b^2 + x^2)^{3/2}}.$$

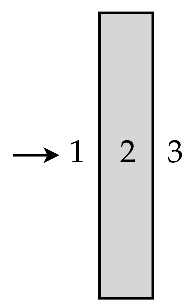
d. Since the solenoid's own B-field is negligible outside it, these two fields dominate at the point in question. Find **S** at that point.

We have $S(x) = \dfrac{E(a)B(x)}{\mu_0} = \dfrac{\pi A^2}{Ra}\dfrac{b^2}{(b^2+x^2)^{3/2}}$.

e. Integrate the flux of **S** over the surface of the solenoid and verify that the answer is the power delivered to the loop by the solenoid.

The flux is $\int \mathbf{S}\cdot d\mathbf{A} = \int_{-\infty}^{\infty} S(x)\cdot 2\pi a\cdot dx = \dfrac{2\pi^2 A^2 b^2}{R}\int_{-\infty}^{\infty}(b^2+x^2)^{-3/2}\,dx$. An integral table gives the integral as $2/b^2$, so we have $\int \mathbf{S}\cdot d\mathbf{A} = \dfrac{(2\pi A)^2}{R}$. This is $\mathcal{E}^2/R = \mathcal{E}I$, the power delivered to the loop, as claimed.

3. Consider light of wavelength λ passing through a glass plate (thickness d, refractive index n) at normal incidence from the left, with air on both sides. In this case there is is an incident wave and a reflected wave in region 1, only a transmitted wave in region 3, but waves going both ways in region 2.



a. Analyze the situation as we did for a single interface, applying the boundary conditions to the fields at the surfaces, to find the transmission coefficient. *Ans*:

$$\frac{1}{T} = 1 + \left[\frac{(n-1)^2}{2n}\sin(nkd)\right]^2,\ \text{where } k = 2\pi/\lambda.$$

The boundary conditions are these. At $x=0$: $E_I + E_R = E_2 + E_2'$, $B_I - B_R = B_2 - B_2'$. At $x=d$: $E_2 e^{inkd} + E_2' e^{-inkd} = E_T$, $B_2 e^{inkd} - B_2' e^{-inkd} = B_T$. In air, $B = E/c$; in glass $B = n\cdot E/c$. This gives four equations for the E's. Eliminating E_2 and E_2', we find after some algebra and using Euler's formula: $\dfrac{E_I}{E_T} = \cos(nkd) - i\dfrac{n^2+1}{2n}\sin(nkd)$. Squaring the magnitude we get $\dfrac{1}{T} = \left|\dfrac{E_I}{E_T}\right|^2 = \cos^2(nkd) + \left(\dfrac{n^2+1}{2n}\right)^2\sin^2(nkd)$. Then $\cos^2\theta = 1 - \sin^2\theta$ gives the answer.

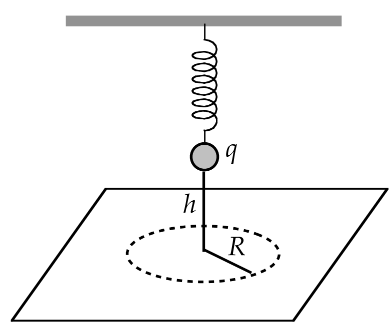
b. What is the shortest wavelength for which T is a minimum? If $n = 1.5$, what is that minimum value of T?

The question should ask for the *longest* wavelength. This is when $nkd = \pi / 2$, or $\lambda = 4nd$. Then $T = \left[1 + \left(\frac{n^2 - 1}{2n}\right)^2\right]^{-1}$. For $n = 1.5$ this gives $T \approx 0.85$. This occurs when there is constructive interference in the reflected wave.

4. The *radiation resistance* is the equivalent resistance that would dissipate the same average power per cycle as a radiating dipole emits in radiation. Show that it is given by $R = (d / \lambda)^2 \cdot 790\ \Omega$, where d is the length of the dipole. [The current in the dipole is $I(t) = (\omega / d) \cdot p(t)$.]

Take the given current and calculate the average of $I^2 R$ over a cycle. We get $P_{av} = \frac{1}{2} \frac{p_0^2 \omega^2 R}{d^2}$. Equate this to the power in Larmor's formula $P_{av} = \frac{p_0^2 \omega^4}{12\pi\varepsilon_0 c^3}$. One finds $R = \frac{(\omega d / c)^2}{6\pi\varepsilon_0 c}$. Using $\omega / c = 2\pi / \lambda$ and the numbers, one finds the answer.

5. A charge q hangs from a spring and oscillates up and down with angular frequency ω as shown, with its equilibrium point at height h above the floor. The amplitude of the oscillation is $d / 2$. Assume $h >> \omega / c >> d$, so the formulas we have derived for dipole radiation can be used.



a What is the intensity of radiation at a point on the circle shown, at distance R from the point directly below the oscillating charge?

The dipole moment is $p(t) = qd\sin\omega t$. The average **S** is given by Eq (11) in the notes:

$\mathbf{S}_{av} = \dfrac{(qd)^2\omega^4}{32\pi^2\varepsilon_0 c^3}\dfrac{\sin^2\theta}{r^2}\hat{\mathbf{r}}$. Here **r** is along the line from the dipole to the point on the circle.

Then $\sin\theta = R/r$. The intensity on the floor is the component of **S** normal to the surface, which is $I(R) = S\cos\theta = S\cdot h/r$. This gives $I(R) = \dfrac{(qd)^2\omega^4}{32\pi^2\varepsilon_0 c^3}\dfrac{R^2 h}{(R^2+h^2)^{5/2}}$.

b. For what value of R is the intensity a maximum?

Setting $dI/dR = 0$ we find $R = \sqrt{\tfrac{2}{3}}h$.