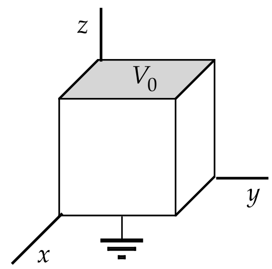
Physics 182

Assignment 2

1. The bottom and sides of a metal cubical box are welded together and grounded. The top of the box is slightly separated and insulated from the rest, and is held at constant potential V_0. Find V at points inside the box. To make the expressions simpler, let each side have length π.



Ans:

$$V(x,y,z)=\frac{16V_0}{\pi^2}\sum_{m=1,3,5,\ldots}\sum_{n=1,3,5,\ldots}\frac{1}{mn}\sin mx\cdot\sin ny\cdot\frac{\sinh\left(\sqrt{m^2+n^2}\cdot z\right)}{\sinh\left(\sqrt{m^2+n^2}\cdot \pi\right)}.$$

The boundary conditions are these:

$$V(0,y,z)=0,\ V(\pi,y,z)=0;$$
$$V(x,0,z)=0,\ V(x,\pi,z)=0;$$
$$V(x,y,0)=0,\ V(x,y,\pi)=V_0.$$

The factors dependent on x and y must be sinusoidal, while the factor dependent on z must be exponential. So we let

$$X(x)=A_1\sin(k_1x+\phi_1),\ Y(y)=A_2\sin(k_2y+\phi_2),\ Z(z)=B_1e^{kz}+B_2e^{-kz},$$

where $k^2=k_1^2+k_2^2$. The boundary conditions in x and y give $\phi_1=\phi_2=0$ and $\sin(k_1\pi)=\sin(k_2\pi)=0$; this means k_1 and k_2 are integers, so we set $k_1=m$ and $k_2=n$. The first boundary condition in z gives $B_1+B_2=0$. The total solution thus has the form $V(x,y,z)=\sum_{m.n}c_{mn}\sin(mx)\cdot\sin(ny)\cdot\sinh(kz)$, where $k=\sqrt{m^2+n^2}$. The remaining boundary condition gives $V_0=V(x,y,\pi)=\sum c_{mn}\sin(mx)\cdot\sin(ny)\cdot\sinh(k\pi)$. Using the orthogonality of the sines (see G, 3.3.1) we can determine the coefficients, giving the result.

2. A non-conducting thin hollow spherical shell of radius R has surface charge density and no charge inside the shell. The charge density is such that the potential on the surface is given by $V_0(\theta) = k\cos(3\theta)$, independent of ϕ.

a. Show that the potential can be written in terms of Legendre polynomials: $V_0(\theta) = \frac{k}{5}[8P_3(\cos\theta) - 3P_1(\cos\theta)]$. Consult G for the polynomials.

> First show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ by using $\cos 3\theta = \cos(\theta + 2\theta)$ and the double angle formulas. Then look up the polynomials to show the result.

b. Show that inside the shell $V(\theta) = \frac{k}{5}\left[8\left(\frac{r}{R}\right)^3 P_3(\cos\theta) - 3\left(\frac{r}{R}\right)P_1(\cos\theta)\right]$.

[Consult G about the orthogonality of the P's.]

> In general for $r \le R$ we have $V(r,\theta) = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)$, so $V_0(\theta) = \sum_\ell A_\ell R^\ell P_\ell(\cos\theta)$. The orthogonality of the P's tells us that $A_3 = \frac{8k}{5R^3}$ and $A_1 = -\frac{3k}{5R}$. All the others vanish.

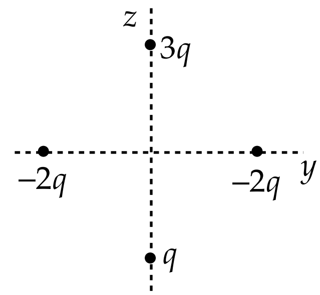
c. Show that outside the shell $V(\theta) = \frac{k}{5}\left[8\left(\frac{R}{r}\right)^4 P_3(\cos\theta) - 3\left(\frac{R}{r}\right)^2 P_1(\cos\theta)\right]$.

> Similarly for $r \ge R$ we have $V(r,\theta) = \sum_\ell B_\ell r^{-(\ell+1)} P_\ell(\cos\theta)$ and $V_0(\theta) = \sum_\ell B_\ell R^{-(\ell+1)} P_\ell(\cos\theta)$. This gives $B_3 = \frac{8}{5}kR^4$ and $B_1 = -\frac{3}{5}kR^2$. Others vanish.

d. Use the relation between the discontinuity in the E-field as you pass through the surface charge, $E_r(\text{out}) - E_r(\text{in}) = \sigma/\varepsilon_0$, to find the charge distribution $\sigma(\theta)$. Here $E_r(\text{out}) = -\left.\frac{\partial V(\text{out})}{\partial r}\right|_{r=R}$, etc.

> Using the given formulas we find $\sigma = \frac{k\varepsilon_0}{5R}[56P_3(\cos\theta) - 9P_1(\cos\theta)]$.

3. Four point charges are arranged as shown, each at distance a from the origin.



a. Find the total charge and dipole moment. [For point charges the dipole moment is given by $\mathbf{p} = \sum_i q_i \mathbf{r}_i$.]

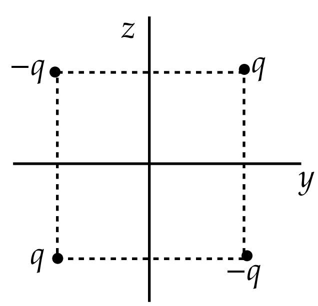
The total charge is zero. The dipole moment is $\mathbf{p} = 3aq\mathbf{k} - aq\mathbf{k} - 2aq\mathbf{j} + 2aq\mathbf{j} = 2aq\mathbf{k}$.

b. Neglecting terms small compared to the dipole term, what is $V(r,\theta,\phi)$?

The dipole term is $V(r,\theta) = k\dfrac{\mathbf{p}\cdot\mathbf{r}}{r^3} = k(2aq)\dfrac{\cos\theta}{r^2}$.

c. Devise an arrangement of four point charges that has neither total charge nor dipole moment, and calculate the elements of the quadrupole moment tensor.

One such arrangement is two dipoles opposite to each other as shown below. Let the square have side $2a$. Then we use $Q_{ij} = \sum q r_i r_j$ to find $Q_{yy} = Q_{zz} = 0$ and $Q_{yz} = Q_{zy} = 4qa^2$. Of course the elements with an x subscript are all zero.



4. The polarization in a sphere of radius R is $\mathbf{P} = \beta\mathbf{r}$, where β is a constant and r is the position vector relative to the center of the sphere. There are no free charges.

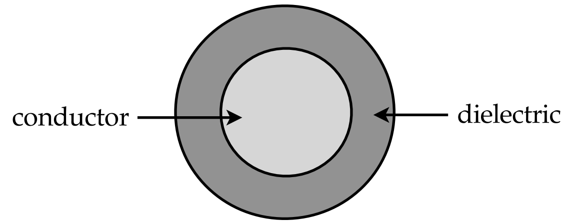
a. Find the bound charge densities σ_{pol} and ρ_{pol}.

The surface bound charge is $\sigma_{pol} = \mathbf{P}\cdot\mathbf{n}$ at the surface. This gives $\sigma_{pol} = \beta R$. The bound charge density is $\rho_{pol} = -\nabla\cdot\mathbf{P}$. This gives $\rho_{pol} = -3\beta$ (since $\nabla\cdot\mathbf{r} = 3$).

b. Find the E-field inside and outside the sphere.

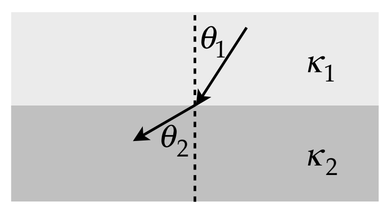
Use Gauss's law. We have $E_r(r) = kQ(r)/r^2$ where $Q(r)$ is the charge enclosed by a sphere of radius r. For $r > R$ all the charge is enclosed. On the surface we have $Q_{surf} = 4\pi R^2\cdot\sigma_{pol} = 4\pi R^3\beta$. For the volume we have $Q_{vol} = \frac{4}{3}\pi R^3\cdot\rho_{pol} = -4\pi R^3\beta$. The total charge is zero, so the field outside is zero (no surprise). For $r < R$ we have only the charge out to radius r, which is $Q(r) = \frac{4}{3}\pi r^3\cdot\rho_{pol} = -4\pi r^3\beta$. The E-field is thus $E_r = -4\pi k\beta\cdot r$. This is directed toward the center of the sphere.

5. A conducting sphere of radius a carries free charge Q. It is surrounded by a dielectric material of susceptibility χ out to radius b as shown. Find the energy in the fields for this configuration.



The E-field for $r > b$ is kQ/r^2 so the energy density is $\eta_e = \frac{1}{8\pi k}E^2 = \frac{kQ^2}{8\pi}\frac{1}{r^4}$. The energy in this part of the field is $U(r > b) = \int_b^\infty dr\, 4\pi r^2 \cdot \eta_e = \frac{kQ^2}{2b}$. Within the dielectric the field is reduced by a factor $1/\kappa = 1/(1+\chi)$. The energy density is $\eta_e = \frac{\kappa}{8\pi k}\left(\frac{kQ}{\kappa r^2}\right)^2 = \frac{kQ^2}{8\pi\kappa}\frac{1}{r^2}$. This gives energy $U(a < r < b) = \frac{kQ^2}{2\kappa}\left(\frac{1}{a} - \frac{1}{b}\right)$. The field is zero for $r < a$. The total energy is thus $U = \frac{kQ^2}{2}\left(\frac{1}{b} + \frac{1}{\kappa a} - \frac{1}{\kappa b}\right)$.

6. At a boundary between two dielectrics the lines of the E-field bend as shown. Show that $\kappa_2 \tan\theta_1 = \kappa_1 \tan\theta_2$, where the κ's are the dielectric constants.



The components of **E** parallel to the surface are continuous, so $E_1 \sin\theta_1 = E_2 \sin\theta_2$. The components of $\kappa\mathbf{E}$ perpendicular to the surface are continuous, so $\kappa_1 E_1 \cos\theta_1 = \kappa_2 E_2 \cos\theta_2$. Dividing these equations gives the result.