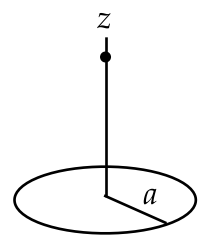
Physics 182

Assignment 1

1. The E-field on an axis of symmetry.

   a. A charge $+Q$ is distributed uniformly around a thin circular ring of radius a, lying in the x-y plane, centered at the origin. Find $\mathbf{E}$ at the point shown on the z-axis. [Use the symmetry: What direction must $\mathbf{E}$ have?] *Ans*:

   $$\mathbf{E} = kQ\frac{z}{\left(a^2+z^2\right)^{3/2}}\mathbf{k}\,.$$



By the symmetry, $\mathbf{E}$ cannot have a component perpendicular to the z-axis. Because the charge is positive, $\mathbf{E}$ must point away from the ring. Consider an infinitesimal bit of the ring of length $ds = a\,d\theta$. This will contain charge $dQ = \frac{Q}{2\pi a}\cdot ds = \frac{Q}{2\pi}\cdot d\theta$. The field from this charge at the field point has magnitude $dE = k\frac{dQ}{r^2}$, where $r^2 = a^2 + z^2$. Its z-component is $dE_z = dE\cdot\cos\alpha = dE\cdot(z/r)$. Putting it all together we have $dE_z = \frac{kQ}{2\pi}\cdot\frac{z}{r^3}\cdot d\theta$. None of the other factors depend on θ, so the integral over that variable gives 2π and we find the result claimed.

   b. Now let this ring have infinitesimal width da and charge per unit area σ. Use the result of (a) and integrate over a to find $\mathbf{E}$ at the point shown when the circle in the drawing represents a uniformly charged disk of radius R. [What is the charge on a ring of radius a, width da, and charge density σ? Find the area of such a ring.] *Ans*: $\mathbf{E} = \frac{2kQ}{R^2}\left[1-\frac{z}{\sqrt{R^2+z^2}}\right]\mathbf{k}$.

The area of the ring is $2\pi a\cdot da$, with charge $2\pi\sigma a\cdot da$. The integral is thus $2\pi k\sigma z\int_0^R da\cdot a\cdot(a^2+z^2)^{-3/2}$. The substitution $\beta = a^2 + z^2$ reduces this to a simple integral. Using $Q = \pi R^2\sigma$ we get the claimed result.

c. Find the potential at the point for the case in (b), assuming $V(\infty)=0$. *Ans*: $V=\frac{2kQ}{R^2}\left(\sqrt{R^2+z^2}-z\right)$.

We have $V(z)=-\int_\infty^z E_z\,dz=-\frac{2kQ}{R^2}\int_\infty^z dz\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)$. Using the same substitution as in (b) we find the claimed result. (To get the lower limit to give zero, use the binomial approximation.)

d. What is the field approximately if the distance to the point is large compared to the radius of the ring in (a), or the disk in (b)?

For $z>>R$ use $(z^2+R^2)^{-1/2}=(1/z)\cdot(1+R^2/z^2)^{-1/2}\approx(1/z)\cdot(1+R^2/2z^2)$ to find $E(z)\approx kQ/z^2$, the field of a point charge. This is a general rule: for great distance from the charges, the E-field is approximately that of a point charge with the total charge (unless the total charge is zero).

e. What is the approximate formula for the field if the radius of the disk in (b) is very large compared to the distance to the point? [This is the "infinite sheet" approximation.]

For $z<<R$ we have easily $E\approx 2kQ/R^2=2\pi k\sigma=\sigma/2\varepsilon_0$, independent of z.

2. The E-field of a spherical distribution of charge is given by $\mathbf{E}=Ar^2\cdot\mathbf{r}$, where A is a positive constant.

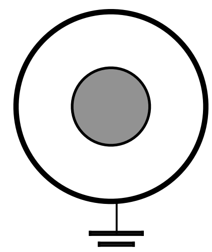
a. Use the differential form of Gauss's law to find the charge density $\rho(r)$. [Look up the divergence in spherical coordinates on the front cover of G.] *Ans*: $\rho=\frac{A}{4\pi k}\cdot 5r^2$.

The r component of $\mathbf{E}$ is Ar^3, so we have $\nabla\cdot\mathbf{E}=\frac{A}{r^2}\frac{\partial}{\partial r}(r^5)=5Ar^2$. This gives the result.

b. How much charge is contained in a sphere of radius *R*? Do it two ways: (1) Integrate your answer to (a) over the volume; (2) Use the integral form of Gauss's law. *Ans*: $Q = \frac{A}{k} R^5$.

The volume integral is $Q = \int_0^R dr \cdot 4\pi r^2 \cdot \rho = \frac{5A}{k} \int_0^R dr \cdot r^4$. This gives the result. The flux integral is $4\pi R^2 \cdot AR^3$. This is equal to $4\pi kQ$, so we get the result immediately.

3. A coaxial cable is a long pair of cylindrical conductors arranged so the cross-section is as shown. The solid inner conductor has radius *a*; the hollow outer conductor has inner radius *b*. On the outer surface of the inner conductor positive charge is spread uniformly in amount λ per unit length of the cable. The outer conductor is grounded.



a. What is the charge per unit length on the outer conductor? How do you know?

There is no charge on the outer surface of the outer conductor because of the grounding. On the inner surface the charge per unit length is $-\lambda$. To show this, consider a Gaussian surface embedded within the material of the outer conductor. The field is zero in the conductor, so the flux through the surface is zero; therefore the enclosed charge is zero.

b. Use Gauss's law in integral form to find the E-field in the gap between the conductors, at distance *r* from the symmetry axis. [Use as your Gaussian surface a cylinder of radius *r* and some length ℓ.] *Ans*: $E = 2k\lambda / r$.

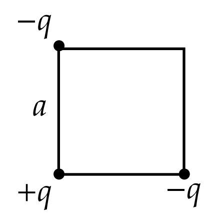
By the symmetry the field lines are radially out from the symmetry axis, therefore parallel to $d\mathbf{A}$ on the curved part of the surface. At the ends $\mathbf{E} \perp d\mathbf{A}$. We have then $\oint \mathbf{E} \cdot d\mathbf{A} = E \cdot 2\pi r \ell$. The enclosed charge is $\lambda\ell$, so Gauss's law gives the result claimed.

c. Find the potential at the inner conductor, assuming the grounding makes the potential on the outer conductor zero. *Ans*: $V(b) = 2k\lambda \ln(a/b)$.

Since $V(a) = 0$ we have $V(b) = -\int_a^b E\,dr = -2k\lambda \int_a^b \frac{dr}{r} = 2k\lambda \ln(a/b)$.

4. Energy stored in a system of point charges.

a. Three point charges are at corners of a square of side a, as shown. How much work must be done by an external agent to bring another charge q from infinite distance and place it at the empty corner? *Ans*: $W = -\frac{kq^2}{a}\left(2 - \frac{1}{\sqrt{2}}\right)$.



The work done is the increase in potential energy, which is the potential at the empty corner multiplied by q. That potential is $-2\frac{kq}{a} + \frac{kq}{\sqrt{2}\cdot a}$.

b. When this charge is in place, what is the total stored energy in this system of four charges? *Ans*: $U = -\frac{kq^2}{a}(4 - \sqrt{2})$.

The potential energy of the three charges shown is $U_1 = -2\frac{kq^2}{a} + \frac{kq^2}{\sqrt{2}\cdot a}$. Adding the amount in (b) we find the result claimed.

5. An isolated conducting sphere of radius R contains charge Q.

a. Find the energy density η_e a a point at distance $r > R$ from the center of the sphere. *Ans*: $\eta_e = \frac{k}{8\pi}\frac{Q^2}{r^4}$.

The E-field is $E(r) = \frac{kQ}{r^2}$, so (using $\eta_e = \frac{1}{2}\varepsilon_0 E^2$ and $\varepsilon_0 = 1/4\pi k$) we have the result.

b. Use this to find the total stored energy in the field of this sphere. *Ans*: $U = \frac{kQ^2}{2R}$.

The energy is $U = \int d^3r\,\eta_e = \int_R^\infty 4\pi r^2\,dr \cdot \frac{kQ^2}{8\pi}\frac{1}{r^4}$. This gives the result. (The integral starts at R because the field is zero inside the sphere.)

c. If we imagine there is a grounded concentric sphere of infinite radius around the given sphere, we have a capacitor. Find its capacitance. *Ans*: $C = R/k$.

For a capacitor $U = Q^2/2C$, so using the result of (c) we get the desired result.