Physics 182

Electrostatics I — Applications

Field lines.

Since fields are themselves invisible, those trying to "visualize" them have resorted to geometric mapping. For a scalar field it is simple enough: one draws curves through points having the same value of the field, making a new curve each time the value increases by a chosen amount. For the potential field these curves are called *equipotentials*. Where they crowd together the potential is changing quickly with location; where they are sparse, it is changing very little.

A vector field presents a different kind of mapping problem: one needs to deal with both the magnitude and direction at each point. One common map is to draw a curve tangent to the direction of the field, attach an arrow to show that direction, and let the "density" of such curves indicate the magnitude. These are *field lines*.

For more detail about the lines of the electrostatic field, and their use in dealing with the integral form of Gauss's law, see G, Sec 2.2. Here we give some general features. Some of these apply only to the E-field set up by static charges.

- Since the E-field is directed away from positive charge and toward negative charge (the directions of the force on a positive test charge) we can say that field lines start on positive charge and end on negative charge. They do not close on themselves.
- As one follows a field line in the direction of the arrow, the potential *falls*. As one moves against the arrow, the potential *rises*. Put another way, as one moves *away* from positive charge, or *toward* negative charge, the potential *falls*, etc.
- Equipotentials cross field lines at right angles.
- Because a conductor is an equipotential region, field lines intersect the surface of a conductor at a right angle.
- Field lines do not penetrate into a conductor, because the E-field is ero within the conductor. (Actually the field does not drop discontinuously to zero, but it becomes negligible over a distance of only a few atomic diameters.)

Field line mapping is not a precise method of calculating anything, but provides some intuition about the field. It has a useful connection to the flux of the field, and to Gauss's law. In a rough sense, the flux through a surface "counts" the number of field lines. In considering Gauss's law, it's easy to see why a positive charge within the volume leads to a positive contribution to the flux: the field lines emanate from the enclosed charge, passing outward through the surface and "measuring" the outward flux. Negative charge within the volume leads to lines passing inward through the surface, and thus to negative flux. Lines from charges not located within the surface pass inward at one part of the surface and out again at another part; the net flux is zero.

Use of Gauss's law to find the field.

In the integral form of Gauss's law, $\oint \mathbf{E} \cdot d\mathbf{A} = 4\pi k Q_{enc}$, **E** is part of an integrand, so this formula is not very useful for calculating **E** directly. But in some cases of high symmetry one can choose the Gaussian surface cleverly enough that **E** will be constant (or zero) and can be pulled through the integral sign. Then one can solve for it.

The most important case is that of spherical symmetry. This means that there is a point (the center of symmetry) about which the physical situation is the same in all directions. An example is a sphere of charge for which the density may depend on the *distance* one goes from the sphere's center, but not on the *direction* goes.

In such a case the lines of the E-field are straight lines emanating radially frm the symmetry center. If the charge is positive, the lines are directed outward from the center; if negative, inward.

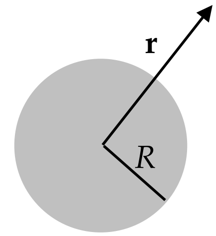
To take advantage of this symmetry in using Gauss's law, one chooses a spherical Gaussian surface. At all points on this sphere **E** will have the same magnitude, E. Since $d\mathbf{A}$ is radially outward, **E** is either parallel (for positive charge) or opposite (for negative charge) to $d\mathbf{A}$. So $\mathbf{E} \cdot d\mathbf{A} = E_r \, dA$, where E_r is the outward radial component of **E** — which is the only component it has, so $E_r = \pm E$. If the Gaussian sphere has radius r then we have $\oint \mathbf{E} \cdot d\mathbf{A} = E_r \int dA = 4\pi r^2 \cdot E_r$.

Call the amount of charge enclosed in this sphere $Q(r)$. Then from Gauss's law we find

$$E_r(r) = k\frac{Q(r)}{r^2}.$$

This is exactly like the field of a single point charge at the origin, with charge equal to $Q(r)$, i.e., to the amount of charge enclosed in a sphere of radius r about the center of symmetry. So the problem is reduced to finding $Q(r)$. *All spherically symmetric situations are like this.*

The symmetry also applies to the potential. The equipotentials are concentric spherical surfaces with centers at the symmetry center.



Example 1. Find the E-field and potential at distance r from the center of a solid non-conducting sphere of radius R in which a total amount of positive charge Q is uniformly distributed.

We use the result above. For $r \geq R$ the charge enclosed is $Q(r) = Q(R) = Q$. Therefore we have

$$E_r = k\frac{Q}{r^2} \text{ for } r \geq R.$$

For points inside the sphere ($r < R$) we must calculate $Q(r)$. Since the charge is uniformly distributed, the ratio $Q(r)/Q$ is simply the ratio of the volumes of shpers of radius r and R, which is r^3 / R^3. So we find

$$E_r = kQ\frac{r}{R^3} \text{ for } r < R.$$

The sign of E_r is the sign of Q, so **E** is outward (inward) for positive (negative) Q.

Now the potential. As is customary, we will choose at infinite distance. We use the definition and the fact that if we choose a path along a radial line then $\mathbf{E} \parallel d\mathbf{r}$. So

$$V(r) = -\int_\infty^r E_r \, dr.$$

Because E_r changes when r becomes less than R, the integral may have two parts. Outside the sphere it is easy:

$$V(r > R) = -kQ\int_\infty^r \frac{1}{r^2} dr = \frac{kQ}{r}.$$

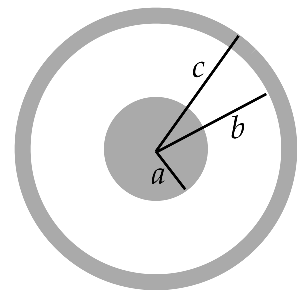
So outside the sphere both the E-field and the potential are just those of a point charge at the origin with charge Q. Inside the sphere we have to break the integral into a part from ∞ to R, plus a part from R to r. The former is just kQ/R, so we have

$$V(r < R) = \frac{kQ}{R} - \frac{kQ}{R^3}\int_R^r r \, dr = \frac{kQ}{2R}\left(3 - \frac{r^2}{R^2}\right).$$

Let us check that the answer gives back $\mathbf{E} = -\nabla V$. For this it is helpful to use spherical coordinates (r, θ, ϕ). (The various vector derivatives are given on the inside front cover of G.) Since V depends only on r, we see that only the radial component of the gradient contributes (as expected, since we know **E** is radially directed). We have $E_r = -\dfrac{\partial V}{\partial r}$.

Outside the sphere we find $E_r = -kQ\frac{\partial}{\partial r}(r^{-1}) = \frac{kQ}{r^2}$. Inside we find

$E_r = \frac{kQ}{2R^3}\frac{\partial}{\partial r}(r^2) = kQ\frac{r}{R^3}$. These are the formulas we obtained from Gauss's law.



<u>Example 2.</u> We have a system of two conductors, a small solid sphere of radius a and a larger spherical shell with inner radius b and outer radius c. The inner sphere carries positive charge Q. The total charge on the shell is zero. Find the E-field and the potential at all distances from the center of symmetry, and find the charges on the two surfaces of the shell.

Because all charges are at rest there is no charge anywhere except on the surfaces of the conductors, and whatever charge is on the inner surface of the shell, equal and opposite charge is on the outer surface. Because of the symmetry the charge on any of these surfaces is distributed uniformly, with the same charge per unit area everywhere on the surface. Finally, the E-field is zero within the inner sphere and between the inner and outer surfaces of the shell.

We use the general formula for spherical symmetry, starting outside the shell ($r > c$). Here $Q(r) = Q$ so we have

$$E_r = \frac{kQ}{r^2} \text{ for } r > c.$$

Of course for $b < r < c$ the field is zero. In the space between the shell and the sphere $Q(r) = Q$ again, so we find

$$E_r = \frac{kQ}{r^2} \text{ for } a < r < b.$$

Again, for $r < a$ the field is zero.

That's the easy part. Outside the shell everything is like a point charge at the origin with charge Q, so we have

$$V(r > c) = \frac{kQ}{r}.$$

From c in to b there is no E-field, so the potential *does not change*. Therefore

$$V(b < r < c) = \frac{kQ}{c}.$$

In the gap between the conductors we have $V(r) = V(b) - \int_b^r E_r\, dr = \dfrac{kQ}{c} - kQ\int_b^r \dfrac{1}{r^2} dr$. This gives

$$V(a < r < b) = kQ\left(\frac{1}{r} - \frac{1}{b} + \frac{1}{c}\right).$$

For $r < a$ the potential remains at it value at *a*, so

$$V(r < a) = kQ\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}\right).$$

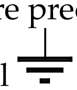
Now about the surface charges. To find the charge on the surface at *b* we use Gauss's law and the fact that the field is zero within the conducing material. Use a Gaussian surface with radius between *b* and *c*. This surface lies entirely within the conductin material, so there is no field and therefore no flux through this surface. By Gauss's law the total enclosed charge must be zero. That charge consists of the charge *Q* on the sphere plus whatever charge is on the surface at *b*. The latter must therefore be $-Q$. It follows that charge on the surface at *c* is $+Q$.

This is easy to visualize in terms of field lines. The lines cannot penetrate the shell, so every line that emanates from the charge at *a* must terminate at *b*. For this to happen the charges on those surfaces must be equal and opposite in sign.

Gauss's law can also be used for cases of axial symmetry, situations in which rotation about a straight line (axis) changes nothing physically. In that case one uses a cylinder around the symmetry axis as a Gaussian surface. See Example 3 of Sec. 2.2 in G.

Grounding and shielding.

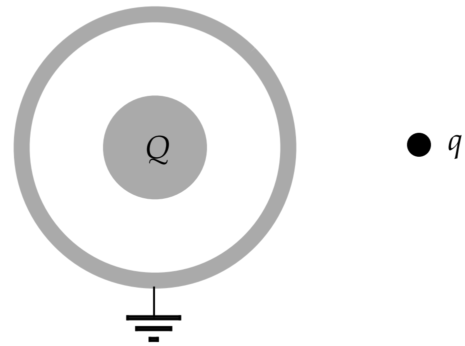
The fact that the static E-field cannot penetrate a conductor has many practical applications. So does the fact that all points on a conductor are at the same potential if all charges are at rest.

The earth is a very large conductor, and any other conductor attached electrically to the earth comes to the potential of the earth. This makes the earth's potential a useful one to take to be zero, measuring all other potential relative to it. Attaching a conductor in a system one is analyzing is called **grounding**. (The British are more precise: they call it "earthing".) The circuit diagram icon for grounding is the symbol  .

Example 3. Suppose the conducting shell in the previous example is grounded. Then its potential will be zero (by choice of the earth to have $V = 0$). How does that happen?

We saw that the charge on the outer surface of the shell (at $r = c$) was $+Q$. If we connect the shell to the earth electrically, that charge will spread out all over the earth (because of the repulsion of like charges), making the amount on the surface of the shell essentially zero. So there are no sources on the outer surface, which means there is no E-field outside the shell, and consequently no change in potential between infinity and the outer surface of the shell. This means the potential is zero everywhere outside the shell.

Now consider the situation if we bring up a point charge q outside the shell, as shown. What happens?



We can give qualitative answers by considering the field lines. Let q be positive. The field lines emanate from it. They must either terminate on negative charges or go off to infinity. The earth will supply some negative charge to put on the outer surface of the shell, which will terminate some of the field lines from q. Because the attractive force is greater at closer distance, most of the negative charge on the outer surface of the shell will accumulate on the side nearest q.

But what changes inside the shell because of the presence of q? Nothing. The field lines from the inner sphere still terminate on the negative charges on the inner surface of the shell. The distribution of charge inside the shell, which resulted in making the potential on the shell zero, will still have that effect. No change is necessary, so none occurs.

In effect, the grounded shell divides the world into two non-interacting parts: inside the shell and outside the shell. This is the phenomenon of **shielding**.

Mathematically this illustrates a uniqueness theorem: if a certain distribution of charges makes the potential at points on a closed surface have certain values, then this is the *only* distribution of charges that can have that effect. In other words, given a certain configuration of conductors, and given the value of the potential at all points on a surface enclosing the system, there is only one possible set of charges on the conductors.

In this case the system is the shell and the sphere inside it, the surface is any surface embedded in the shell, on which we know the potential is zero everywhere because the conductor is grounded. Without the external charge we know how the charges inside the shell arrange themselves. The presence of the external charge does not change the potential on the embedded surface. Therefore its presence does not change the distribution of charge on anything inside the shell.

The same theorem can be applied to the system consisting of the outer surface of the shell and the world outside that. If we hold q fixed and move the inner sphere with charge Q to a different location in the cavity within the shell, there will be no effect on the charge distribution on the outer surface of the shell, and no change in the force exerted on q by the shell. (There will be changes in the region inside of the shell: the field in the cavity will no longer be spherically symmetric, nor will the charge distributions on the inner sphere and the inner surface of the shell.) The grounded conductor truly divides the world into non-interacting parts.

Capacitors as energy storage devices.

Because the electric field possesses energy, one can make energy storage devices in which the energy is located in an electric field. The most common such device is a capacitor. This consists of two conductors separated from each other. Charge is put on each of them, usually equal and opposite charge so the field is strongest in the region between them.

Because of this E-field, there is a potential difference between the conductors, which we will call ΔV. Let the charge on the positive conductor be Q. Then the **capacitance** of the system (a measure of its effectiveness in storing energy) is defined by

$$C = Q / \Delta V .$$

Let us find the energy stored. Suppose charge q is already on the positive conductor, and $-q$ on the negative conductor, while the potential difference is ΔV. To increase the charge we move an amount dq from the negative to the positive conductor. Let the potential of the negative plate be held to zero (perhaps by grounding it). Then the removal of dq from it makes no change in the potential energy, but adding dq to the positive conductor at potential ΔV increases the potential energy by the amount $dq \cdot \Delta V$. This is therefore the amount of work that must be done by some external agent (perhaps a battery) to move the charge. In any case, we have an increase in potential energy $dU = dq \cdot \Delta V$. But $\Delta V = q / C$, so we have

$$dU = \frac{1}{C} q\, dq .$$

If we define $U = 0$ when there is no charge on the conductors, then the potential energy when the positive one has charge Q is given by integrating this expression:

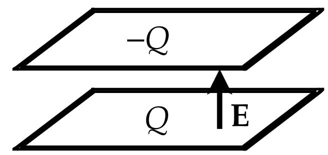
$$U = \frac{Q^2}{2C}.$$

Alternatively, using $Q = C\Delta V$, we have

$$U = \tfrac{1}{2} C \cdot \Delta V^2 .$$

Since in general ΔV is proportional to Q, C is determined only by the nature of the configuration of conductors — and, as we will see, the properties of the space around them.

<u>Example 4.</u> In a few cases of simple geometry one can calculate C easily. In introductory courses one usually works out the case of two parallel flat plates, as shown. Let the plates have area A and be separated by distance d. Let them be charged as shown. What is ΔV ? The simplest approach is to find **E** first.



One usually makes some simplifying assumptions. If d is small compared to the dimensions of the plates, then between them (where E is largest by far) one uses the field of an infinite plate, which one can get from Gauss's law (see Examples 4 and 5 in Chap 2 of G): $E = \sigma / \varepsilon_0$, where $\sigma = Q / A$ is the area charge density on the positive plate. Then we have, by integrating over a path from the negative to the positive plate: $\Delta V = Ed = (Q / A\varepsilon_0) \cdot d$. Then $C = Q / \Delta V$ gives $C = \varepsilon_0 A / d$.

This is an approximation, of course. It assumes that **E** is uniform in the region between the plates, and that it drops to zero discontinuously at the edge of that region. Neither is strictly true, and the second is a violation of the field equation $\nabla \times \mathbf{E} = 0$. But it serves the pedagogical purpose.

In this approximation it is easy to calculate the stored energy from the energy density formula $\eta_e = \frac{1}{2}\varepsilon_0 E^2$. We simply integrate and use the "fact" that E is uniform between the plates and zero elsewhere:

$$U = \int \eta_e d^3 r = \eta_e \cdot (Ad) = \tfrac{1}{2}\varepsilon_0 (\Delta V / d)^2 \cdot (Ad) = \tfrac{1}{2}(\varepsilon_0 A / d) \cdot \Delta V^2 = \tfrac{1}{2} C \cdot \Delta V^2 .$$

In principle one can always find the stored energy in the E-field this way, but it's not usually this easy.

Capacitors have more uses in electronic circuits than as simple storage devices. But their fundamental principle is always that an electric field possesses energy.