Exam I

Solutions

In questions or problems not requiring numerical answers, express the answers in terms of the symbols for the quantities given, and standard constants such as g. In numerical questions or problems, use g = 10 m/s².

Part A: Choose the best answer; 4 points each.

1. A particle is moving with constant acceleration.

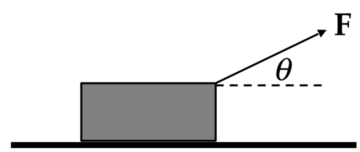
   A. It must move in a straight line. [Projectile motion]

   ☛B. The total force on it must be constant.

   C. Its speed must change at a constant rate. [Projectile motion]

   D. The distance it moves in time t is proportional to t^2. [Projectile motion]

2. A box of mass m is being pulled across a rough floor by application of a force **F** as shown. The force of kinetic friction acting on the box is:



   A. $\mu_k mg$.

   B. Unknown without knowing the block's speed..

   C. $\mu_k F\sin\theta$.

   ☛D. $\mu_k(mg - F\sin\theta)$.

3. A car goes over the top of a hill, the road being the arc of a vertical circle of radius R. The maximum speed v the car can have and not leave the road:

   A, Is Rg.

   B. Is $\sqrt{R/g}$.

   ☛C. Is $\sqrt{Rg}$. [If the normal force is zero, $mg = mv^2/R$.]

   D. Depends on the mass of the car.

4. Which of the following statements about energy and power is NOT true?

   A. The power input by force **F** to an object moving with velocity **v** is given by $P = \mathbf{F} \cdot \mathbf{v}$.

   ☛B. The total work done by conservative forces is equal to the change in kinetic energy. [All forces.]

   C. All constant forces are conservative.

   ☛D. Non-conservative forces always reduce the total energy. [An exploding bomb.]

   [Both B and D are correct answers.]

Part B: Choose T or F, depending on whether the statement is true or false; 3 points each.

1. Taking into account air resistance (which, like kinetic friction, opposes the motion) a ball thrown straight up in the air takes longer to come down than to reach its maximum height. **T** [The average speed is less.]

2. A person standing on a scale in an elevator that is accelerating upward at 0.2 g will observe the scale to read 80% of the person's normal weight. **F** [120%.]

3. When a skydiver of mass m is falling with constant terminal speed v, the power input by the air resistance force is $-mgv$. **T** [Equal and opposite to power input by gravity.]

Part C: Problems. Indicate your method clearly. An answer supported by no argument, or a fallacious one, may receive little or no credit. Problems have the point values shown.

1. A ball is thrown into the air at elevation angle θ with initial speed v_0. The equation of its trajectory in the usual x-y axes is $y = x\tan\theta - \dfrac{gx^2}{2v_0^2\cos^2\theta}$.

   a. At what value of x does the ball reach its maximum height?

   b. Show that the maximum height is $h = \dfrac{v_0^2}{2g}\sin^2\theta$.

   c. Suppose the ball lands at vertical distance d below the level from which it was thrown. Show that its speed on landing is given by $v_f^2 = v_0^2 + 2gd$, regardless of the value of θ. [You may use energy considerations.]

[15 points]

a. To find the maximum height, set $dy/dx = 0$. We find $0 = \tan\theta - \dfrac{gx}{v_0^2\cos^2\theta}$, or

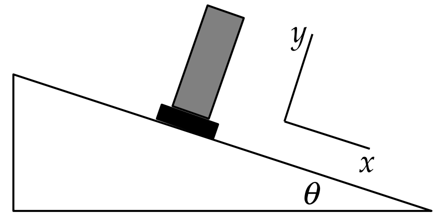
$$x = \frac{v_0^2\sin\theta\cos\theta}{g}.$$

b. Substituting this value of x into the trajectory equation, we find (after some algebra) the claimed result.

c. Use $v^2 = v_0^2 + 2\mathbf{a}\cdot(\mathbf{r}-\mathbf{r}_0)$. With y positive upward, $\mathbf{a} = -g\mathbf{j}$, so $\mathbf{a}\cdot(\mathbf{r}-\mathbf{r}_0) = -g(y-y_0)$. Use $y = -d$ and $y_0 = 0$, which leads to the result claimed.

Or use conservation of energy. The initial energy (taking gravitational potential energy to be zero at the original height) is $\frac{1}{2}mv_0^2$. The final energy is $\frac{1}{2}mv^2 - mgd$, so we find $\frac{1}{2}mv^2 = \frac{1}{2}mv_o^2 + mgd$. Rearranging gives the claimed result.

2-1. A crate of mass m is resting on a scale, and the two objects slide together down a frictionless incline as shown. [Use axes shown.]



a. Show that $\mathbf{g}_{eff}$ in the frame of the crate is perpendicular to the incline. [What is the acceleration of crate and scale?]

b. What weight does the scale read for the crate, in terms of m, g, and θ?

[10 points]

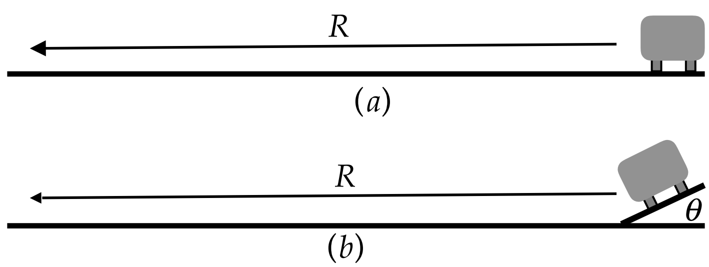
a. The acceleration is $a = g\sin\theta$ in the $+x$-direction. Write $\mathbf{g}_{eff} = \mathbf{g} - \mathbf{a}$ in components: $(g_{eff})_x = g\sin\theta - g\sin\theta = 0$; $(g_{eff})_y = -g\cos\theta$. This proves the claim.

b. It reads $mg_{eff} = mg\cos\theta$.

2-2. A car is rounding a curve of radius R at speed v. In (a) the road is level; in (b) the road is banked. The car moves into the page at the instant shown.

c. For (a), what is the minimum static friction coefficient μ_s that will allow the car to make the curve without sliding?

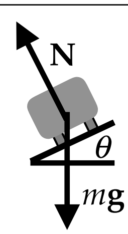
d. For (b), what is the banking angle θ that will allow the car to make the curve without needing to use friction, because the horizontal component of the normal force supplies the radial acceleration. [Draw a free-body diagram.]



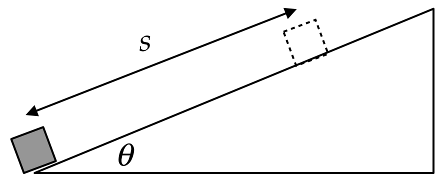
[10 points]

c. Friction provides the radial acceleration, so $f_s = mv^2/R$. If it is the maximum friction force, then $f_s = \mu_s N = \mu_s mg$, so $\mu_s mg = mv^2/R$, or $\mu_s = v^2/Rg$.

d. Diagram below. We have $N\sin\theta = mv^2/R$ and $N\cos\theta - mg = 0$. Eliminating N we get $\tan\theta = v^2/Rg$.



3a. A block of mass m starts with speed v_0 at the bottom of an incline. It slides up distance s before stopping momentarily. The coefficient of kinetic friction is μ_k. Give answers in terms of the given quantities and g



a. What is s in terms of the other quantities? [What is the work done by friction?]

b. If static friction cannot hold it and it slides back down, what is its speed v when it returns to the bottom?

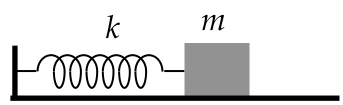
[10 points]

a. The friction force is $f_k = \mu_k N = \mu_k mg\cos\theta$. The work it does during the trip up the incline is $W_f = -f_k \cdot s = -\mu_k mgs\cos\theta$. The final total energy (with gravitational potential energy zero at the bottom) is $E_2 = mgh = mgs\sin\theta$. The initial total energy is $E_1 = \frac{1}{2}mv_0{}^2$. From $W_f = E_2 - E_1$ we have $-\mu_k mgs\cos\theta = mgs\sin\theta - \frac{1}{2}mv_0{}^2$. Solving this we find $s = \dfrac{v_0{}^2}{2g(\sin\theta + \mu_k\cos\theta)}$.

b. The work done by friction on the way down is the same (force and displacement both change directions). The final energy is $E_2 = \frac{1}{2}mv^2$ and the initial energy is $E_1 = mgs\sin\theta$. So we have $-\mu_k gs\cos\theta = \frac{1}{2}mv^2 - mgs\sin\theta$. Substituting from (a) for s, this gives $v^2 = 2gs(\sin\theta - \mu_k\cos\theta) = v_0{}^2\dfrac{\sin\theta - \mu_k\cos\theta}{\sin\theta + \mu_k\cos\theta}$.

[This can also be done using the kinematics equations for constant acceleration, once the acceleration down the incline is found.]

3b. Later we will find that a block, attached to a spring and sliding on a frictionless surface as shown, moves according to the equation $x(t) = A\cos\omega t$, where $\omega = \sqrt{k/m}$ and A is the maximum distance the spring is stretched or compressed.



.a. For this situation, show that Newton's 2nd law can be written as $-kx = m\dfrac{d^2x}{dt^2}$, and prove that the expression given satisfies this equation.

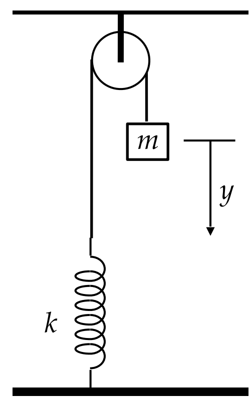
b. Find the velocity and calculate the kinetic energy. then add it to the potential energy $U = \frac{1}{2}kx^2$. Use the formula given for ω and prove that the total mechanical energy is constant in time.

[10 points]

a. The force is $F=-dU/dx=-kx$, and by definition $a=d^2x/dt^2$, so $F=ma$ gives the equation claimed. Doing the derivatives, we find $a=-\omega^2 A\cos(\omega t)$. Using $\omega^2=k/m$, we see that the equation is satisfied.

b. The velocity is $v=dx/dt=-\omega A\sin(\omega t)$. The kinetic energy is $K=\frac{1}{2}mv^2=\frac{1}{2}m\omega^2A^2\sin^2(\omega t)=\frac{1}{2}kA^2\sin^2(\omega t)$. The potential energy is $U=\frac{1}{2}kx^2=\frac{1}{2}kA^2\cos^2(\omega t)$, so $E=K+U=\frac{1}{2}kA^2$, independent of time.

4. A block of mass m is attached by a string over an ideal pulley to a spring of stiffness k. Initially the block is held at rest as shown with the spring unstretched but the string taut. The block is released and falls.



a. Take gravitational potential energy to be zero with the block at its original height, express the total potential energy $U(y)$ of the system (including the spring) as a function of the distance y through which the block has fallen (y is positive *downward*).

b. At what value of y does the block have its maximum kinetic energy? What is the maximum kinetic energy? [What is the *total* mechanical energy of the system?]

c. At what value of y does the block stop (momentarily) before rising again?

d. Describe the subsequent motion of the block.

[20 points]

a. For the block, $U_{grav}=-mgy$ (because y is positive downward); for the spring, $U_{spring}=\frac{1}{2}ky^2$, so $U_{tot}=\frac{1}{2}ky^2-mgy$.

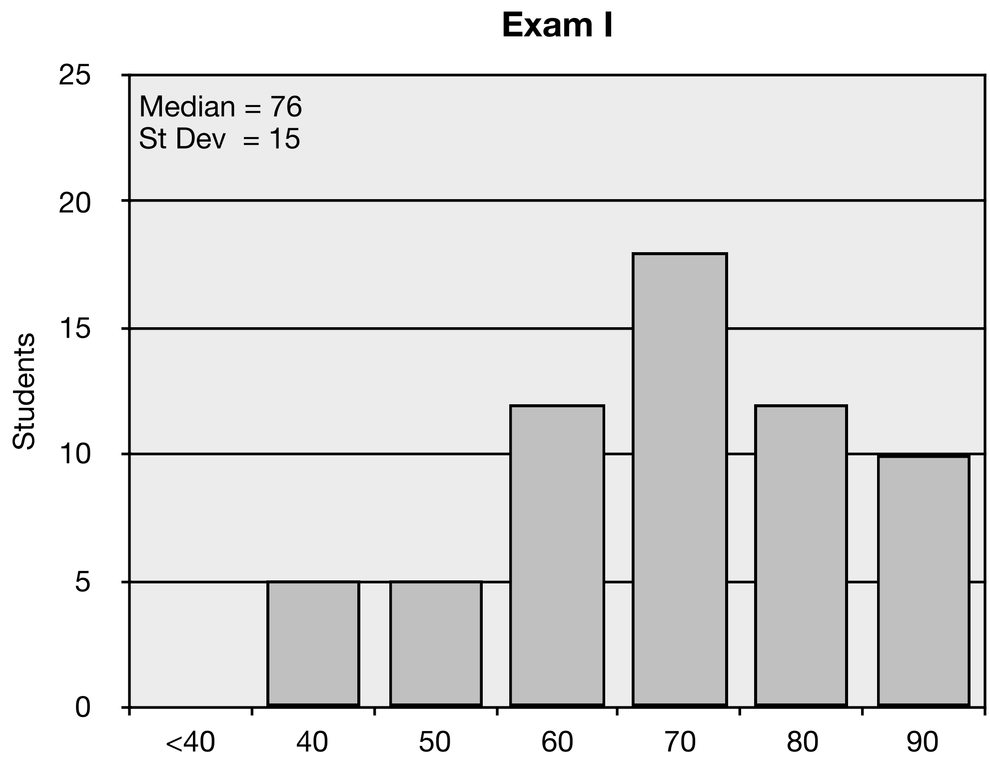
b. When U_{tot} is a minimum, kinetic energy will be a maximum. Set $dU_{tot}/dy=0$: $ky-mg=0$, so the value is $y=mg/k$. Since the block is initially at rest, with zero potential energy, the total energy is zero, so $K=-U$. At the point in question,

$$K=mg(mg/k)-\tfrac{1}{2}k(mg/k)^2=\frac{(mg)^2}{2k}.$$

c. Kinetic energy will be zero again when potential energy is also zero, so set $U_{tot}=0$. We find $\frac{1}{2}ky^2-mgy=0$. The non-zero root is $y=2mg/k$.

d. It oscillates (down and up) between $y=0$ and $y=2mg/k$.

[The fact that the spring is attached to the floor and the string runs over the pulley is irrelevant. The motion of the block is the same as if the spring had been attached to the ceiling.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