PHY 302 HW 1 Classical Mechanics

$\boxed{\text{Solutions}}$:

1. $$\frac{G m M(R)}{R^2} = \frac{m v^2}{R}$$

$$\Rightarrow M(R) = \frac{R v^2}{G}$$

$$M(10\,\text{kpc}) = 1.045 \times 10^{11}\, M_{sun}$$

$$M(20\,\text{kpc}) = 2.09 \times 10^{11}\, M_{sun}$$

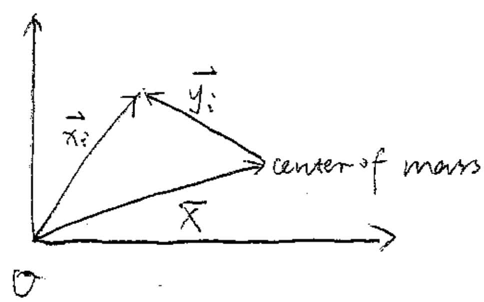
$$M(30\,\text{kpc}) = 3.135 \times 10^{11}\, M_{sun}$$

$(v \approx 150\ \text{km/s})$

2. The total angular momentum is

$$\vec{L} = \sum_i m_i \vec{x}_i \wedge \dot{\vec{x}}_i$$

The total angular momentum about the center of mass is



$$\vec{L}_c = \sum_i m \vec{y}_i \wedge \dot{\vec{y}}_i\,,$$

where $\vec{y}_i = \vec{x}_i - \vec{X}$

$$\Rightarrow \vec{L} = \sum_i m_i \vec{x}_i \wedge \dot{\vec{x}}_i$$

$$= \sum_i m_i (\vec{y}_i + \vec{X}) \wedge (\dot{\vec{X}} + \dot{\vec{y}}_i)$$

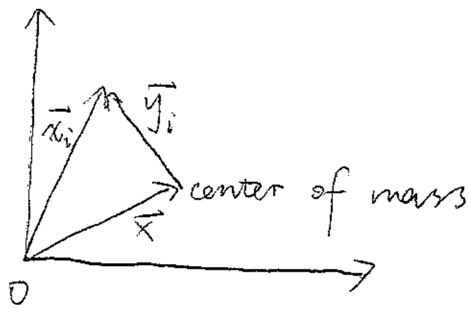
$$= \sum_i m_i \vec{X} \wedge \dot{\vec{X}} + \sum_i m_i \vec{y}_i \wedge \dot{\vec{y}}_i + \overbrace{\sum_i m_i \vec{y}_i}^{0} \wedge \dot{\vec{X}} + \vec{X} \wedge \frac{d}{dt} \overbrace{\sum_i m_i \vec{y}_i}^{0}$$

$$= M \vec{X} \wedge \dot{\vec{X}} + \vec{L}_c$$

Got the two parts.

3. $T = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} \sum_i m_i \dot{y}_i^2$

where $\vec{y}_i = \vec{x}_i - \vec{X}$



Continuous :

$$T = \frac{1}{2}\left(\int \rho(x) d^3x\right) \dot{X}^2 + \frac{1}{2}\int \rho(x) \dot{y}_x^2 \, d^3x$$

4. $$\frac{dT}{dt} = \frac{d\left(\frac{1}{2} m \vec{v}^2\right)}{dt} = m\vec{v} \cdot \frac{d\vec{v}}{dt} = m\vec{a} \cdot \vec{v} = \vec{F} \cdot \vec{v}$$

$$\frac{d(mT)}{dt} = \frac{d\left(m \cdot \frac{1}{2} m v^2\right)}{dt} = \vec{p} \cdot \frac{d\vec{p}}{dt} = \vec{F} \cdot \vec{p}$$

5. $a_{11} x^1 x^1 + a_{12} x^1 x^2 + a_{21} x^2 x^1 + a_{22} x^2 x^2 = 1$

Define $a_{11} = a$, $a_{12} + a_{21} = b$, $a_{22} = c$, $x^1 = x$, $x^2 = y$

$\Rightarrow ax^2 + bxy + cy^2 = 1$

$\Rightarrow 2ax\dot{x} + bx\dot{y} + b\dot{x}y + 2cy\dot{y} = 0$

$$\Rightarrow \frac{\dot{y}}{\dot{x}} = -\frac{by + 2ax}{bx + 2cy} = m$$

i.e. $\dot{y} = m\dot{x}$

$\dot{x}^2 + \dot{y}^2 = v^2$

$$\Rightarrow \begin{cases} \dot{x} = \dfrac{v}{\sqrt{1+m^2}} = v_x \\ \dot{y} = \dfrac{mv}{\sqrt{1+m^2}} = v_y \end{cases}$$

$$\dot{x}^2 + \dot{y}^2 = v^2 \Rightarrow 2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} = 0 \Rightarrow \frac{\ddot{y}}{\ddot{x}} = -\frac{\dot{x}}{\dot{y}} = \frac{bx + 2cy}{by + 2ax}$$

$$\Rightarrow \ddot{y} = \frac{bx + 2cy}{by + 2ax} \ddot{x}, \quad \text{i.e. } a_y = \frac{bx + 2cy}{by + 2ax} a_x$$

for $t = \Delta t$, the position of the particle is $(x + \Delta t\, v_x,\ y + \Delta t\, v_y)$

$$\frac{\dot{y}}{\dot{x}} = -\frac{b(y + v_y \Delta t) + 2a(x + v_x \Delta t)}{b\cdot(x + v_x \Delta t) + 2c(y + v_y \Delta t)} = m'$$

$$\Rightarrow \begin{cases} v_x' = \dfrac{v}{\sqrt{1+m'^2}} \\ v_y' = \dfrac{v}{\sqrt{1+m'^2}} \end{cases}$$

$$a_x = \lim_{\Delta t \to 0} \frac{v_x' - v_x}{\Delta t}$$

$$= -\frac{1}{2} \cdot \frac{v}{(1+m^2)^{3/2}} \cdot 2m \cdot \left[- \frac{(b^2 - 4ac)(x\dot{y} - \dot{x}y)}{(bx + 2cy)^2} \right]$$

$$= \frac{-(b^2 - 4ac)\cdot mv \cdot (v_x y - x v_y)}{(1+m^2)^{3/2}\,(bx + 2cy)^2}$$

$$= \frac{(4ac - b^2)(y - mx)\cdot mv^2}{(1+m^2)^2\,(bx + 2cy)^2}$$

$$= \frac{2(b^2 - 4ac)(by + 2ax)\,v^2}{[(bx + 2cy)^2 + (by + 2ax)^2]^2}$$

$$a_y = \frac{bx + 2cy}{by + 2ax}\, a_x$$

$$= \frac{2(b^2 - 4ac)(bx + 2cy)\,v^2}{[(bx + 2cy)^2 + (by + 2ax)^2]^2}$$

6. $y_1 = x_1 \cos wt - x_2 \sin wt$

$\dot{y}_1 = \dot{x}_1 \cos wt - w x_1 \sin wt - \dot{x}_2 \sin wt - w x_2 \cos wt$

$\ddot{y}_1 = \ddot{x}_1 \cos wt - 2\dot{x}_1 w \sin wt - w^2 x_1 \cos wt - \ddot{x}_2 \sin wt$
$\quad - 2\dot{x}_2 w \cos wt + w^2 x_2 \sin wt$

$y_2 = x_1 \sin wt + x_2 \cos wt$

$\dot{y}_2 = \dot{x}_1 \sin wt + w x_1 \cos wt + \dot{x}_2 \cos wt - w x_2 \sin wt$

$\ddot{y}_2 = \ddot{x}_1 \sin wt + 2\dot{x}_1 w \cos wt - w^2 x_1 \sin wt + \ddot{x}_2 \cos wt$
$\quad - 2\dot{x}_2 w \sin wt - w^2 x_2 \cos wt$

when $\ddot{x} = 0$,

$$\left.\begin{aligned} \ddot{y}_1 &= \underbrace{-2w(\dot{x}_1 \sin wt + \dot{x}_2 \cos wt)}_{①} - \underbrace{w^2 y_1}_{②} \\ \ddot{y}_2 &= \underbrace{-2w(\dot{x}_2 \sin wt - \dot{x}_1 \cos wt)}_{①} \underbrace{- w^2 y_2}_{②} \end{aligned}\right\} \ddot{y} \text{ does NOT vanish.}$$

Part ① : coriolis acceleration

Part ② : centripetal acceleration

CM #1.5

Solution: First notice that, for a general ellipse

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

$$B^2-4AC<0.$$

(a) $a_{11}x'x'+a_{12}x'x^2+a_{21}x^2x'+a_{22}x^2x^2=1$

Define $a_{11}=a.\quad a_{12}+a_{21}=b.\quad a_{22}=c$

$x'=x.\quad x^2=y$

$\Rightarrow\quad ax^2+bxy+cy^2=1\quad$ ①

Differentiate ① $\Rightarrow\quad 2ax\dot{x}+bx\dot{y}+b\dot{x}y+2cy\dot{y}=0.$

$\Rightarrow\quad \dot{x}(2ax+by)=-\dot{y}(bx+2cy)\quad$ ②

Now. ~~tt~~ I argue that $2ax+by\neq 0$.

prove it: if $2ax+by=0$. Since $a\neq 0\quad x=-\frac{by}{2a}$

$\Rightarrow a\frac{b^2}{4a^2}y^2+b(-\frac{b}{2a}y)y+cy^2=y^2(\frac{b^2-4ac}{4a})<0$

which is not consitent with ①

$\Rightarrow\quad 2ax+by\neq 0.$

$bx+2cy=0\quad y=-\frac{b}{2c}x\quad \ldots=\frac{1}{4c}(-b^2+4ac)x^2>0$ ✓

back to ②, define $-\frac{bx+2cy}{2ax+by}=\gamma$, where $2ax+by\neq 0$

$\Leftarrow bx+2cy\neq 0$

$\Rightarrow\quad \dot{x}=\gamma\dot{y}.\quad$ ⑤

$\because\quad \dot{x}^2+\dot{y}^2=v^2$ ③ $\Rightarrow\quad (1+\gamma^2)\dot{y}^2=v^2\quad\Rightarrow \dot{y}=\frac{v}{\sqrt{1+\gamma^2}}$

$\dot{x}=\frac{\gamma v}{\sqrt{1+\gamma^2}}$ ④

Differciate ③. $2\dot{x}\ddot{x}+2\dot{y}\ddot{y}=0\quad\Rightarrow\quad \dot{x}\ddot{x}=-\dot{y}\ddot{y}$

$\Rightarrow\quad \ddot{x}\gamma\dot{y}=-\dot{y}\ddot{y}\quad\Rightarrow\quad \ddot{y}=-\gamma\ddot{x}$

Diff ⑤ $\quad\ddot{x}=\gamma\ddot{y}+\dot{\gamma}\dot{y}=-\gamma^2\ddot{x}+\dot{\gamma}\dot{y}$

$\Rightarrow\quad \ddot{x}=\frac{\dot{\gamma}}{1+\gamma^2}\dot{y}=\frac{v}{(1+\gamma^2)^{\frac{3}{2}}}\dot{\gamma}\quad$ and $\ddot{y}=-\gamma\ddot{x}$

$$\gamma = -\frac{bx+2cy}{2ax+by}$$

$$\dot{\gamma} = -\frac{(b\dot{x}+2c\dot{y})(2ax+by)-(2a\dot{x}+b\dot{y})(bx+2cy)}{(2ax+by)^2}$$

$$= -\frac{2abx\dot{x}+2bcy\dot{y}+4acx\dot{y}+b^2y\dot{x}-2abx\dot{x}-2bcy\dot{y}-b^2x\dot{y}-4ac\dot{x}y}{(2ax+by)^2}$$

$$= -\frac{(4ac-b^2)}{(2ax+by)^2}\left[x\dot{y}-y\dot{x}\right]$$

$$= -\frac{(4ac-b^2)}{(2ax+by)^2}\left[x\frac{v}{(1+\gamma^2)^{\frac{1}{2}}}-y\frac{\gamma v}{(1+\gamma^2)^{\frac{1}{2}}}\right]$$

$$= -\frac{(4ac-b^2)}{(2ax+by)^2}\frac{V}{\sqrt{1+\gamma^2}}(x-\gamma y)$$

$$= -\frac{(4ac-b^2)}{(2ax+by)^2}\frac{V}{\sqrt{1+\gamma^2}}\left[x+\frac{(bx+2cy)y}{2ax+by}\right]$$

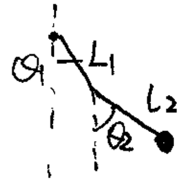
$$= \left[\frac{2ax^2+2bxy+2cy^2}{2ax+by}\right] \quad (=2)$$

$$= -\frac{(4ac-b^2)}{(2ax+by)^3}\frac{2V}{\sqrt{1+\gamma^2}}$$

$$\Rightarrow \ddot{x} = \frac{-2V^2}{(1+\gamma^2)^2}\frac{(4ac-b^2)}{(2ax+by)^3} = \frac{(b^2-4ac)V^2(x-y\gamma)}{(2ax+by)^2(1+\gamma^2)}$$

$$\ddot{y} = -\gamma\ddot{x} = \frac{(4ac-b^2)\gamma V^2(x-\gamma y)}{(2ax+by)^2(1+\gamma^2)}$$

Classical Mechanics HW2



$x_1 = l_1 \cos\theta_1$ $\quad$ $x_2 = l_1\cos\theta_1 + l_2\cos\theta_2$ $\quad$ $\dot{x}_2 = -l_1 \sin\theta_1 \dot\theta_1 - l_2 \sin\theta_2 \dot\theta_2$

$y_1 = l_1 \sin\theta_1$ $\quad$ $y_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$ $\quad$ $\dot{y}_2 = l_1\cos\theta_1\dot\theta_1 + l_2 \cos\theta_2 \dot\theta_2$

(1) $T = \frac{1}{2} m_1 l_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 (l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_2 - \theta_1))$

$V = - m_1 g l_1 \cos\theta - m_2 g (l_1 \cos\theta + l_2 \cos\theta_2)$ $\quad$ $\dot{x}_2^2 + \dot{y}_2^2 = 2 l_1 l_2 \dot\theta_1 \dot\theta_2 [\cos(\theta_1)\cos\theta_2 + \sin\theta_1 \sin\theta_2$

$$\begin{cases} (m_1+m_2) l_1 \ddot\theta_1 + \underbrace{m_2 l_2 \ddot\theta_2 (\sin(\theta_1-\theta_2) + \cos(\theta_1 - \theta_2)) + (m_1+m_2) g \sin\theta_1}_{?} = 0 \\ l_2 \ddot\theta_2 + l_1 \ddot\theta_1 \cos(\theta_1-\theta_2) - l_1 \dot\theta_1^2 \sin(\theta_1-\theta_2) - g \sin\theta_2 = 0 \end{cases}$$

(2) $T = \frac{1}{2} m (r^2 \dot\theta^2 + \dot r^2)$

$V = - m g r \cos\theta + \frac{1}{2} k (r - L)^2$

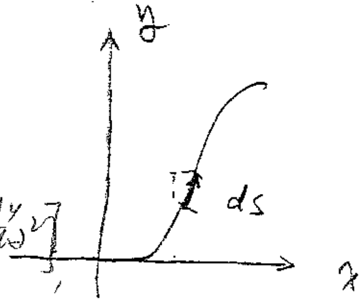
$$\begin{cases} m\ddot r - m r \dot\theta^2 - m g \cos\theta + k(r-L) = 0 \\ r\ddot\theta + 2\dot r \dot\theta + g \sin\theta = 0 \end{cases}$$

(3) $ds = dx \sqrt{1 + (\frac{dy}{dx})^2}$

System

$T = \frac{1}{2} m x^2 w^2 + \frac{1}{2} m \dot s^2$ $\quad = \frac{1}{2} m (\dot x^2 + \dot y^2) = \frac{1}{2}\dot x^2 [1 + (\frac{dy}{dx})^2] = \frac{1}{2}(d\dot s)^2$

$= \frac{1}{2} m x^2 w^2 + \frac{1}{2} m (1 + A^2 n^2 x^{2n-2}) \dot x^2$



$V = mgA|x^n|$

$m(1 + A^2 n^2 x^{2n-2}) \ddot x - m w^2 x - m A^2 n^2 (n-1) \dot x x^{2n-3} + n m g A x^{n-1} = 0$

Equilibrium $\quad \dot x|_{x_0} = \ddot x|_{x_0} = 0$

$x_0 = 0, \quad x_0 = (\frac{w^2}{ngA})^{\frac{1}{n-1}}$

for $n=2$. $\quad w^2 = 2gA$

④

$$L' = \frac{1}{2}mv^2 - q\left(\phi - \frac{1}{c}\frac{\partial \psi}{\partial t}\right) + \frac{q}{c}\left(\vec{A} + \nabla\psi\right)\cdot\vec{v}$$

$$= \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\vec{A}\cdot\vec{v} + \frac{q}{c}\left(\frac{\partial\psi}{\partial t} + \nabla\psi\cdot\vec{v}\right)$$

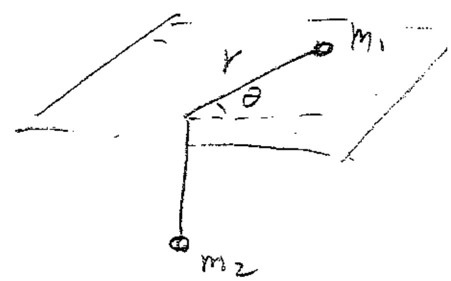
$$= L + \frac{q}{c}\frac{d}{dt}\psi$$

Lagrange changes. But L' still satisfies the equation of motion.

⑤

$$T = \frac{1}{2}m_1\left(\dot{r}^2 + (l-r)^2\dot{\theta}^2\right) + \frac{1}{2}m_2\dot{r}^2$$

$$V = -m_2 g r.$$



$$\begin{cases} (m_1+m_2)\ddot{r} + m_2 g - m_1 r\dot{\theta}^2 = 0. \\ m_1 r\left(r\ddot{\theta} + 2\dot{\theta}\dot{r}\right) = 0 \end{cases}$$

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0.$$

$$r^2\dot{\theta} = L \quad (\text{const.})$$

⑥

$$L = \frac{1}{2}mB^2 + (2mC+F)Bt + (2mC^2+FC)t^2$$

$$S = \int_0^{t_0} L\,dt = \frac{1}{2}mB^2 t_0 + (2mC+F)B\frac{t_0^2}{2} + (2mC^2+FC)\frac{t_0^3}{3}$$

$$\frac{dS}{dC} = 0. \quad \& \quad A = 0. \quad \Rightarrow \quad C = \frac{F}{2m} \qquad B = \frac{a}{t_0} - \frac{Ft_0}{2m}$$

⑦ constraints: $f_1 = l - r - R = 0.$

$$\boxed{f_2 = R\theta - r\varphi + r\theta = 0.}$$

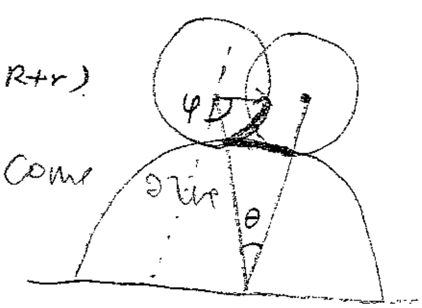
$l \neq R + r$

$l\theta = r\varphi$?

$$L = \frac{1}{2}m\left(\dot{l}^2 + l^2\dot{\theta}^2 + r^2\dot{\varphi}^2\right) - mgl\cos\theta. \quad ; \quad L = L(\theta, \varphi, l).$$

$$\begin{cases} m\ddot{l} - ml\dot{\theta}^2 + mg\cos\theta = \lambda_1 \\ ml^2\ddot{\theta} + \dot{\theta}2ml\dot{l} + mgl\sin\theta = \boxed{\lambda_1} + \lambda_2(R+r) \\ mr^2\ddot{\varphi} = -\lambda_2 r \end{cases}$$

How it come



$$\Rightarrow \quad \cos\theta = \frac{1}{2} \qquad \theta = 60°.$$

CM#2.1

Solution: $\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)\right]$
$+ m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = \frac{\partial\mathcal{L}}{\partial\theta_1} \Rightarrow m_1 l_1^2\ddot{\theta}_1 + m_2\left[l_1^2\ddot{\theta}_1 + 2l_1 l_2\ddot{\theta}_2\cos(\theta_2-\theta_1) - 2l_1 l_2\dot{\theta}_2\sin(\theta_2-\theta_1)(\dot{\theta}_2-\dot{\theta}_1)\right] = -m g l_1\sin\theta_1 - m_2 g l_1\sin\theta_1$$

$$\Rightarrow m_1 l_1\ddot{\theta}_1 + m_2 l_1\ddot{\theta}_1 + \cancel{2}m_2 l_2\ddot{\theta}_2\cos(\theta_2-\theta_1) - \cancel{2}m_2 l_2\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1) + m_1 g\sin\theta_1 + m_2 g\sin\theta_1 = 0$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = \frac{\partial\mathcal{L}}{\partial\theta_2} \Rightarrow m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_2-\theta_1) - m_2 l_1 l_2\dot{\theta}_1(\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1) = -m_2 g l_2\sin\theta_2$$

$$\Rightarrow l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - l_1\dot{\theta}_1(\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1) + g\sin\theta_2 = 0.$$

CM#2.2

Solution: $\mathcal{L} = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta - \frac{1}{2}K(l-r)^2$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{\partial\mathcal{L}}{\partial r} \Rightarrow m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta + K(l-r)$$

$$\Rightarrow m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta - K(l-r) = 0$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{\partial\mathcal{L}}{\partial\theta} \Rightarrow \dot{(mr^2\dot{\theta})} = -mgr\sin\theta$$

$$\Rightarrow 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + mgr\sin\theta = 0$$

$$\Rightarrow 2\dot{r}\dot{\theta} + r\ddot{\theta} + g\sin\theta = 0.$$

CM #23

Solution: $y = A|x^n|$. $\Rightarrow$ $y = A r^n$

$\Rightarrow T = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$

$\dot{x}_1^2 + \dot{x}_2^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + r^2\omega^2$

$\Rightarrow T = \frac{1}{2} m(\dot{r}^2 + r^2\omega^2 + A^2 n^2 r^{2n-2}\dot{r}^2)$

$V = +mgy = mgAr^n$

$\Rightarrow L = \frac{1}{2} m\dot{r}^2(1 + A^2n^2r^{2n-2}) + \frac{1}{2} mr^2\omega^2 - mgAr^n$

$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \Rightarrow m(1 + A^2n^2r^{2n-2})\ddot{r} + m\dot{r}(A^2n^2(2n-2)r^{2n-3}\dot{r})$

$= +mr\omega^2 - mgAnr^{n-1}$

$\Rightarrow m(1 + A^2n^2r^{2n-2})\ddot{r} - mr\omega^2 + m\dot{r}(A^2n^2)(2n-2)r^{2n-3}\dot{r}$

$- \cancel{mr\omega^2} + mgAnr^{n-1} = 0$

Equilbum

$\dot{r} = \ddot{r} = 0$. $\quad mr\omega^2 = mgAnr^{n-1}$

$\Rightarrow r^{n-2} = \left(\frac{mr\omega^2}{mgA}\right) \quad \Rightarrow r^{n-2} = \frac{\omega^2}{gAn}$

for $n=2$ $\quad \omega^2 = 2gA$

CM #2.6

Solution

2.5

2.6 $\mathcal{L} = T - V$.

$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m [2ct + B]^2$

$V = -Fx = -F(A + Bt + ct^2)$

$\Rightarrow L = \frac{1}{2} m [4c^2t^2 + B^2 + 4Bct] + F(A + Bt + ct^2)$

$\Rightarrow S = \int_{x=0}^{x=a} L\,dt = \int_0^t L\,dt$

$= \int_0^t \frac{1}{2} m\{4c^2t^2 + B^2 + 4Bct] + F[A + Bt + ct^2]\,dt$

$m\ddot{l} - ml\dot{\theta}^2 + mg\cos\theta = \lambda_1$

$ml^2\ddot{\theta} + {}_2$

$$\begin{cases} m\ddot{\rho} + mg\cos\theta - m\boxed{\rho}\dot{\theta}^2 = \boxed{\lambda_1} \\ m\rho^2\ddot{\theta} + m\rho\dot{\rho}\dot{\theta} + mg\sin\theta\,\rho = \boxed{\lambda_2}(R + r) \\ mr^2\boxed{\ddot{\phi}} = -\lambda_2 r \\ \rho = R + r \\ \rho\theta = r\varphi \end{cases}$$

Known Known

$\Rightarrow \theta = ?$

$\rho\dot{\theta} = r\dot{\varphi}$ $\rho\ddot{\theta} = r\ddot{\varphi}$

$$\begin{cases} mg\cos\theta - m\rho\dot{\theta}^2 = \lambda_1 \\ m\rho^2\ddot{\theta} + mg\rho\sin\theta = \lambda_2(R + r) \\ m\rho^2 \frac{r}{R+r}\ddot{\theta} = -\lambda_2 r \end{cases}$$

No φ.

7. Solution:

$f_1 = \rho - r - R = 0 \qquad f_2 = (R+r)\theta - r\varphi = 0.$

$\lambda_1 \qquad \lambda_2$

$$L = T - V = \frac{1}{2} m[\dot{\rho}^2 + \rho^2\dot{\theta}^2 + r^2\dot{\phi}^2] - mg\rho\cos\theta$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial\dot{\rho}} = \frac{\partial L}{\partial\rho} + \sum\lambda_i\frac{\partial f_i}{\partial\rho} \Rightarrow m\ddot{\rho} - m\rho\dot{\theta}^2 + mg\cos\theta = \lambda_1 \quad ①$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = \sum\lambda_i\frac{\partial f_i}{\partial\theta} \Rightarrow m\rho^2\ddot{\theta} + m\rho\dot{\rho}\dot{\theta} + mg\rho\sin\theta = \lambda_2(R+r) \quad ②$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} - \frac{\partial L}{\partial\phi} = \sum\lambda_i\frac{\partial f_i}{\partial\phi} \Rightarrow mr^2\ddot{\phi} = -\lambda_2 r \quad ③$$

$$f_1 = 0 \Rightarrow \rho = R+r \quad ④$$

$$f_2 = 0 \Rightarrow (R+r)\theta = r\varphi \quad ⑤.$$

$$⑤ \Rightarrow \ddot{\varphi} = \frac{(R+r)}{r}\ddot{\theta}$$

$$④ \Rightarrow \dot{\rho} = 0. \quad \ddot{\rho} = 0$$

$$\Rightarrow -m\rho\dot{\theta}^2 + mg\cos\theta = \lambda_1 \quad ⑥$$

$$m\rho^2\ddot{\theta} + mg\rho\sin\theta = \lambda_2(R+r) \quad ⑦$$

$$\cancel{m\frac{(R+r)^2}{r}\ddot{\theta} = -\lambda_2 r}$$

$$\Rightarrow m(R+r)^2\ddot{\theta} + mg\sin\theta(R+r) = -m(R+r)^2\ddot{\theta}$$

$$mr(R+r)\ddot{\theta} = -\lambda_2 r \quad ⑧$$

$$\Rightarrow -(R+r)\ddot{\theta} = +g\sin\theta$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{R+r}\sin\theta$$

$$\Rightarrow \dot{\theta}^2 = \frac{g}{R+r}\cos\theta \Rightarrow ⑥$$

$$\Rightarrow 2mg\cos\theta + mg = \lambda_1 = 0$$

$$\Rightarrow \cos\theta = \frac{1}{2}$$

HW3.

3

1 the constraint equation:

$$f(\rho, z, \phi) = z - \rho^2/a = 0$$

~~the~~ Lagrangian:

$$L = \frac{1}{2} m (\dot\rho^2 + \rho^2 \dot\phi^2 + \dot z^2) - mgz$$

introduce Lagrangian multiplier λ. the Lagrange's Eqn.

$$m\ddot\rho = m\rho\dot\phi^2 - 2\lambda\rho/a$$

$$m\ddot z = -mg + \lambda \qquad (*)$$

$$m(\rho^2\ddot\phi + 2\rho\dot\rho\dot\phi) = 0$$

Integrate the equations.

$$m\rho^2\dot\phi = l$$

$$\frac{1}{2} m\dot\rho^2 = -\frac{l^2}{2m\rho^2} - \lambda\rho^2/a + c_1$$

$$\frac{1}{2} m\dot z^2 = (-mg + \lambda) z + c_2$$

where l, c_1, c_2 are constants

$$\frac{1}{2} m (\dot\rho^2 + \dot z^2) = \frac{1}{2} m \left(1 + \frac{a}{4z}\right)\dot z^2 = -\frac{l^2}{2maz} - mgz + E.$$

$$m\ddot z = -mg + \frac{(E + mga/4)\, a + 2l^2/ma}{4\left(z + \frac{a}{4}\right)^2} \qquad (*)$$

Compare (*) & (*)

$$\lambda = \frac{(E + mga/4)\, a + 2l^2/ma}{4\left(z + \frac{a}{4}\right)^2}$$

For max/min values of height. $\dot z \big|_{z_0} = 0.$

$$-\frac{l^2}{2maz_0} - mgz_0 + E = 0$$

$$z_0 = \frac{E}{2mg}\left[1 \pm \left(1 - \frac{2gl^2}{aE^2}\right)^{1/2}\right]$$

$$E = \frac{1}{2} m (\dot\rho^2 + \dot z^2 + \rho^2\dot\phi^2) + mgz$$

$$= \frac{1}{2} m \left(\dot\rho^2 + \dot z^2 + \rho^2 \cdot \frac{l^2}{m^2\rho^4}\right) + mgz \qquad \text{Solve } z$$

$$= \frac{1}{2} m \left(\dot\rho^2 + \dot z^2 + \frac{l^2}{m^2 a z}\right) + mgz = \frac{l^2}{2maz} + mgz \le E$$

2. the constraint equations

$$f_1(\rho, z, \phi) = \rho - b = 0$$
$$f_2(\rho, z, \phi) = z - a\phi = 0.$$

Lagrangian.

$$L = \frac{1}{2} m(\dot\rho^2 + \dot z^2 + \rho^2 \dot\phi^2) - \frac{1}{2} k(\rho^2 + z^2)$$

introduce λ_1, λ_2. as Lagrange multipliers

Lagrange's Eqn:

$$m\ddot\rho = m\rho\dot\phi^2 - k\rho + \lambda_1$$
$$m\ddot z = -kz + \lambda_2$$
$$m(\rho^2\ddot\phi + 2\rho\dot\rho\dot\phi) = -a\lambda_2.$$

integrate.

$$\frac{1}{2} m\dot z^2 = -k\left(\frac{a^2}{a^2+b^2}\right)\frac{z^2}{2} + \frac{1}{2} m\dot z_0^2$$

we have

$\dot z_0 = \dot z|_{z=0}$ constant

$$\lambda_1 = \left(1 - \frac{m\dot z_0^2}{ka^2} + \frac{z^2}{a^2+b^2}\right) kb.$$

$$\lambda_2 = \frac{b^2}{a^2+b^2} kz.$$

3. transformation $(\rho, z, \theta) \to (\rho, z - k\epsilon, \theta + \epsilon)$

under the transformation

$$L = \frac{1}{2} m(\dot\rho^2 + \dot z^2 + \rho^2\dot\theta^2) - V(\rho, k\theta + z).$$

clearly L does not depend on ϵ.

$$Q = \sum \frac{\partial L}{\partial \dot q_r} \frac{\partial q_r}{\partial \lambda}\Big|_{\lambda=0} = m\rho^2\dot\theta - mk\dot z$$

is conserved.

4. Lagrangian

$$L = \frac{1}{2} m[(a\dot{\theta})^2 + (\omega a \cos\theta)^2] - mga\sin\theta.$$

Lagrange's Eqn.

$$ma\ddot{\theta} = -\frac{1}{2} m\omega^2 a^2 \sin(2\theta) - mga\cos\theta.$$

Integrate,

$$\frac{1}{2} ma^2\dot{\theta}^2 = \frac{1}{2} m\omega^2 a^2 \cos^2\theta - mga\sin\theta + E.$$

Constant of motion is Energy E

$$E = \frac{1}{2} ma^2\dot{\theta}^2 + mga\sin\theta - \frac{1}{2} m\omega^2 a^2 \cos^2\theta$$

Stationary point θ_0: $\dot{\theta}|_{\theta_0} = \ddot{\theta}|_{\theta_0} = 0.$

$$\left(\sin\theta_0 + \frac{g}{\omega^2 a}\right)\cos\theta_0 = 0.$$

nontrivial solution only when $\left|\frac{g}{\omega^2 a}\right| < 1$, or $\omega_c = \frac{g}{a}$.

5. $L = \frac{1}{2} m\dot{\vec{r}}^2 + V(\vec{r})$.

$$G = L + \lambda\sigma(\vec{r}, t).$$

Lagrange Eqn.

$$\frac{d}{dt}\left(\frac{\partial G}{\partial \dot{\vec{r}}}\right) - \frac{\partial G}{\partial \vec{r}} = 0$$

consider.
$$\frac{dG}{dt} = \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{\vec{r}}}\right)\dot{\vec{r}} + \frac{\partial G}{\partial \dot{\vec{r}}}\frac{d\dot{\vec{r}}}{dt} + \frac{\partial G}{\partial t}$$

$H = p\dot{q} - L$

$$\frac{d}{dt}\left(\frac{\partial G}{\partial \dot{\vec{r}}}\dot{\vec{r}} - G\right) + \frac{\partial G}{\partial t} = 0.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}}\dot{\vec{r}} - L\right) + \lambda\frac{\partial \sigma}{\partial t} = 0.$$

$$E = H = \frac{\partial L}{\partial \dot{\vec{r}}}\dot{\vec{r}} - L.$$

$$\frac{d\sigma}{dt} = \frac{\partial \sigma}{\partial \vec{r}}\dot{\vec{r}} + \frac{\partial \sigma}{\partial t} = 0$$

$\frac{d}{dt}E = \{E, H\} + \frac{\partial E}{\partial t}$

$$\frac{dE}{dt} = -\lambda\frac{\partial \sigma}{\partial t}.$$

which is not zero.

energy is not conserved

$$-\lambda\frac{\partial \sigma}{\partial \vec{r}}\cdot\dot{\vec{r}}$$

6. Lagrangian.

$$L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) - V(\vec{r}) + \frac{e}{2} B r^2 \dot{\theta}$$

conjugate momenta are.

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + \frac{e}{2} B r^2$$

Hamiltonian

$$H = \dot{r} p_r + \dot{\theta} p_\theta - L = \frac{p_r^2}{2m} + \frac{(p_\theta - \frac{e}{2} B r^2)^2}{2mr^2} + V(\vec{r})$$

In rotating coordinate.

$$\omega = -\frac{eB}{2m}.$$

$$r' = r, \quad \theta' = \theta - \omega t.$$

$$L' = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}'^2) - V(\vec{r}) + \frac{e^2 B^2 r^2}{8m}.$$

$$p_r' = m\dot{r}$$

$$p_\theta' = mr^2\dot{\theta}'$$

$$H' = \dot{r} p_r + \dot{\theta}' p_\theta' - L'$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta'^2}{2mr^2} + V(\vec{r}) - \frac{e^2 B^2 r^2}{8m}.$$

HW 4.

4

1. (a) A system is Hamiltonian if there exists H satisfying

$$\begin{cases} \frac{\partial H}{\partial p_i} = \dot{q}_i \\ \frac{\partial H}{\partial q_i} = -\dot{p}_i \end{cases} \quad \text{or.} \quad \frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} = 0.$$

In the given system.

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{p}_x}{\partial p_x} = 0 + 0 = 0$$

$$\frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{p}_y}{\partial p_y} = -2ax + 2ax = 0.$$

thus the System is Hamiltonian.

(b). Construct H.

$$\frac{\partial H}{\partial p_x} = \dot{x} = p_x - ay^2. \qquad H = \frac{p_x^2}{2} - ay^2 p_x + f_1(x, y, p_y)$$

$$\frac{\partial H}{\partial x} = -\dot{p}_x = kx - 2ay(p_y - 2axy) \qquad H = \frac{kx^2}{2} - 2axy\,p_y + 2a^2y^2x^2 + f_2(y, p_x, p_y)$$

$$\frac{\partial H}{\partial p_y} = \dot{y} = p_y - 2axy \qquad H = \frac{p_y^2}{2} - 2axy\,p_y + f_3(x, y, p_x)$$

$$\frac{\partial H}{\partial y} = -\dot{p}_y = -ky - 2ax(p_y - 2axy) - 2ay(p_x - ay^2) \qquad H = -\frac{ky^2}{2} - 2axy\,p_y + 2a^2x^2y^2 - ap_xy^2 + a^2\frac{y^4}{2} + f_4(x, p_x, p_y)$$

combine the results

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{k}{2}(x^2 - y^2) - 2axy\,p_y + 2a^2x^2y^2 - ay^2 p_x + \frac{a^2}{2}y^4.$$

2 (cont.).

$$\begin{cases} P_1 = \frac{\partial F}{\partial q_1} = 2p_1 q_1 + p_2 - (q_1 + q_2)^2 \\ P_2 = \frac{\partial F}{\partial q_2} = p_2 = -(q_1 + q_2)^2 \end{cases}$$

$$\begin{cases} p_1 = \frac{P_1 - P_2}{2q_1} \\ p_2 = P_2 + (q_1 + q_2)^2 \end{cases}$$

c). $K = p_1^2 + p_2$

$\dot{Q}_1 = \frac{\partial k}{\partial p_1} = 2p_1$ $\qquad Q_1 = 2p_1 t + a$

$\dot{Q}_2 = \frac{\partial k}{\partial p_2} = 1$ $\qquad Q_2 = t + b$

$\dot{p}_1 = -\frac{\partial k}{\partial Q_1} = 0.$ $\qquad p_1 = \text{const}$

$\dot{p}_2 = -\frac{\partial k}{\partial Q_2} = 0.$ $\qquad p_2 = \text{const}.$

$q_1 = \sqrt{Q_1} = \sqrt{2p_1 t + a}$

$q_2 = Q_2 - \sqrt{Q_1} = t + b - \sqrt{2p_1 t + a}$

$P_1 = 2p_1 q_1 + p_2 - (q_1 + q_2)^2 = 2p_1\sqrt{2p_1 t + a} + p_2 - (t+b)$

$P_2 = -(t+b)^2.$

(a) choose type 2 generating function.

$$\frac{\partial F_2}{\partial P_1} = Q_1 = q_1^2 \Rightarrow F_2 = q_1^2 P_1 + f(q_1, q_2, P_2)$$

$$\frac{\partial F_2}{\partial P_2} = Q_2 = q_1 + q_2 \Rightarrow F_2 = q_1^2 P_1 + (q_1 + q_2) P_2 + g(q_1, q_2)$$

(b) $K = H = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2$

and we have,

$$p_1 = \frac{\partial F}{\partial q_1} = 2P_1 q_1 + P_2 + \frac{\partial g(q, t)}{\partial q_1}$$

$$p_2 = \frac{\partial F}{\partial q_2} = P_2 + \frac{\partial g(q, t)}{\partial q_2}$$

$$q_1 = \sqrt{Q_1}$$

$$q_2 = Q_2 - \sqrt{Q_1}$$

$$K = \left(\frac{2P_1 q_1 + \frac{\partial g}{\partial q_1} - \frac{\partial g}{\partial q_2}}{2q_1}\right)^2 + P_2 + \frac{\partial g}{\partial q_2} + Q_2^2$$

K only depend on P_1, P_2 we have.

$$\frac{\partial g}{\partial q_1} - \frac{\partial g}{\partial q_2} = 0. \qquad \frac{\partial g}{\partial q_2} = -Q_2^2.$$

$$\frac{\partial g}{\partial q_2} = -(q_1 + q_2)^2 = -q_1^2 - q_2^2 - 2q_1 q_2.$$

$$g = -q_1^2 q_2 - \frac{1}{3} q_2^3 - q_1 q_2^2 + h(q_1)$$

$$\frac{\partial g}{\partial q_1} = -2q_1 q_2 - q_2^2 + \frac{\partial h(q_1)}{\partial q_1} = -q_1^2 - q_2^2 - 2q_1 q_2 = \frac{\partial g}{\partial q_2}$$

$$h(q_1) = -\frac{1}{3} q_1^3$$

$$g(q_1, q_2) = -q_1^2 q_2 - q_1 q_2^2 - \frac{1}{3} q_2^3 - \frac{1}{3} q_1^3$$

$$= -\frac{1}{3}(q_1 + q_2)^3$$

i) $F = P_1 q_1^2 + P_2 (q_1 + q_2) - \frac{1}{3}(q_1 + q_2)^3$

ii) $K = P_1^2 + P_2$

Consider.

$$\frac{d}{dt}\{f,g\} = \{\{f,g\},H\} + \frac{\partial}{\partial t}\{f,g\}$$

use Jacobi's Identity

$$\{\{f,g\},H\} = -\{\{g,H\},f\} - \{\{H,f\},g\}$$

$$\frac{\partial}{\partial t}\{f,g\} = \{\frac{\partial f}{\partial t}, g\} + \{f, \frac{\partial g}{\partial t}\}$$

$$\frac{d}{dt}\{f,g\} = \{\frac{\partial f}{\partial t} + \{f,H\}, g\} + \{f, \frac{\partial g}{\partial t} + \{g,H\}\}$$

$$= \{\frac{df}{dt}, g\} + \{f, \frac{dg}{dt}\} = 0.$$

thus $\{f,g\}$ is a constant of motion

4. $H = q_1 p_1 - q_2 p_2 - aq_1^2 + bq_2^2$

$$\dot{f}_1 = \{f_1, H\} = -\frac{p_2 - bq_2}{q_1^2} q_1 + \frac{b}{q_1} q_2 - \frac{1}{q_1}(2bq_2 - q_2) = 0$$

$$\dot{f}_2 = \{f_2, H\} = q_2 q_1 + q_1(-q_2) = 0$$

$$\dot{f}_3 = \{f_3, H\} = e^{-t} q_1 - q_1 e^{-t} = 0$$

H is not depend on t explicit. f_1, f_2, f_3 are constant of the motion. $\Rightarrow$ H is also a constant of motion

5. a) $H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right)$

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} = pq^4 \\ \dot{p} = -\frac{\partial H}{\partial q} = \frac{1}{q^3} - 2p^2 q^3 \end{cases} \Rightarrow q\ddot{q} - 2\dot{q}^2 - q^2 = 0.$$

the eq. of motion for q.

b) Consider a canonical transformation.

$$\begin{cases} Q = -\frac{1}{q} \\ P = pq^2 \end{cases}$$ then we have $K = \frac{1}{2}Q^2 + \frac{1}{2}P^2$

6. $\dot{D} = \frac{1}{2}(p\dot{q} + q\dot{p}) - H = \frac{1}{2}\left(p(p) - q\frac{1}{q^2}\right) - \frac{p^2}{2} + \frac{1}{2q^2} = 0.$

In general case

$$H = (p_i p^i)^{n/2} - a\,(q_i q^i)^{-n/2}$$

$$D = \frac{p_i q^i}{n} - Ht.$$

$$\dot{D} = \frac{p_i \dot{q}^i + \dot{p}_i q^i}{n} - H.$$

$$= \frac{1}{n}\left(p_i \frac{\partial H}{\partial p^i} - q^i \frac{\partial H}{\partial q^i}\right) - H = \frac{1}{n}(nH) - H = 0$$