Physics 302
Final Exam

17 December 2010, 2:00-5:00 PM

The Exam is out of 90 points. There are 10 bonus points. Show all your work for partial credit. There are a total of 8 problems.

1. The canonical coordinate part of a canonical transformation is

$$Q = \arctan(\lambda q/p)$$

where $\lambda \neq 0$. Complete the canonical transformation by showing that the conjugate momentum

$$P(q, p, t) = \frac{1}{2}(p^2/\lambda + \lambda q^2) + R(q, p, t)$$

where R is an arbitrary polynomial function of (q/p). (10 points)

2. A particle of mass m is subjected to the force $F = -kq - \alpha/q^3$.

(a) Show that a possible Hamiltonian for this system is

$$H = p^2/(2m) + \frac{1}{2}kq^2 + Ap/q$$

where A is a properly chosen constant. (10 points)

(b) Use the canonical transformation of Problem 1 to solve for the motion. (10 points)

(c) Discuss the relation of this Hamiltonian H to the Hamiltonian H' that would be found by simply finding a potential $V(q)$ from the given F, constructing the Lagrangian $L' = T - V$, and then obtaining the Hamiltonian by the usual procedure. (10 points)

3. A system with one degree of freedom consists of a bead of mass m sliding on a frictionless V-shaped wire which is standing upright in a uniform gravitational field. Assume no energy loss as the bead passes through the lowest point of the V where the two straight sections of the wire meet. Write down the Lagrangian and the Hamiltonian for this bead. Choose the coordinate q to be the horizontal component of the bead's position vector. Solve for the motion to obtain $p(t)$ and $q(t)$ in the Hamiltonian formulation. (10 points)

4. Use the action-angle variables to find how the period depends on the total energy of the bead in Problem 3. (10 points)

5. The initial angle of the straight sections of the wire in Problem 3 is $45°$ with respect to the horizontal. The initial period of oscillation of the bead is T_0. The angle of the wires is very slowly increased until the wires are vertical. Find the new period of the oscillation. (10 points)

6. Show that a one-dimensional particle subject to the force

$$F = -kx^{2n+1}$$

where n is an integer, will oscillate with a period proportional to A^{-n} where A is the amplitude. Pay special attention to the case of $n \leq 0$. *Hint: use a dimensionality-based argument.* (10 points)

7. A particle is projected vertically upward to a height h above a point on the earth's surface at a latitude λ. Take the acceleration due to gravity to be g, and the earth's angular velocity to be ω. Neglect air resistance, and assume h is small. Find the distance between the point of origin and the point of impact on the earth. Is this horizontal deflection in the east, west, north or south direction from the point of origin? (10 points)

8. A circular hoop is made of a thin wire such that the mass of the hoop is M and the radius is R. Define a Cartesian coordinate system such that the origin is at the center of the hoop, and the x and y axes define the plane of the hoop. Write the formal expressions (as integrals) for the three principal moments of inertia of the hoop, I_{xx}, I_{yy} and I_{zz}, and evaluate them in terms of M and R (Hint: think about it instead of using brute force!). (10 points)