Homework Assignment 4
Physics 302, Classical Mechanics

Fall, 2010

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Handed out:	Friday, September 24, 2010
Due in class on:	Friday, October 1, 2010

Problems

(Ten points for each problem unless noted otherwise.)

1. (15 *points*)
   Consider the following first-order differential equations of motion of a particle in an external field:

$$\begin{aligned}
\frac{dx}{dt} &= p_x - ay^2 \\
\frac{dp_x}{dt} &= -kx + 2ay(p_y - 2axy) \\
\frac{dy}{dt} &= p_y - 2axy \\
\frac{dp_y}{dt} &= ky + 2ax(p_y - 2axy) + 2ay(p_x - ay^2),
\end{aligned}$$

   where a and k are nonzero constants.

   (a) Show that this system is Hamiltonian without constructing the Hamiltonian.

   (b) Construct the Hamiltonian for this Hamiltonian flow.

2. (20 *points*)
   Consider the following Hamiltonian

$$H = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2.$$

   To solve this problem, we will make a canonical transformation so that new coordinates are related to the old as

$$Q_1 = q_1^2, \quad Q_2 = q_1 + q_2.$$

   (a) Construct a generic generating function for this transformation;

   (b) Find a particular generating function which will transform this Hamiltonian to a new Hamiltonian which will depend only on P_1 and P_2, i.e. Q_1 and Q_2 are cyclic (missing) variables. To document your work,

      i. write down this particular generating function;

      ii. write down the new Hamiltonian K;

      iii. write down the other two relations in the canonical transformation, $P_1 = P_1(q, p)$, and $P_2 = P_2(q, p)$.

   (c) Solve this problem to obtain expressions for q_1, q_2, p_1, and p_2 as functions of time and initial values.

3. Prove Poisson's theorem: if f and g are constants of the motion for a Hamiltonian system, then $\{f, g\}$ is a constant of the motion.

4. Consider the following Hamiltonian

$$H = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2,$$

where a and b are constants. Show that the following dynamical variables are constants of the motion:

$$f_1 = (p_2 - b q_2)/q_1, \quad f_2 = q_1 q_2, \quad f_3 = q_1 e^{-t}.$$

Do there exist other independent constants of the motion? If they do exist, find them.

5. The Hamiltonian for a system has the form

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right)$$

Find the equation of motion for q.

Find a canonical transformation that reduces H to the form of a harmonic oscillator. Show that the solution for the transformed variables is such that the equation of motion found for q is satisfied.

6. For a one-dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2}$$

show that there is a constant of the motion

$$D = \frac{pq}{2} - Ht$$

As a generalization of the above, for motion in a plane with the Hamiltonian

$$H = |\vec{p}|^n - a r^{-n}$$

where $\vec{p}$ is the vector of the momenta conjugate to the Cartesian coordinates, show that there is a constant of the motion

$$D = \frac{\vec{p} \cdot \vec{r}}{n} - Ht$$