

# Web Appendix 2

## PLANCK'S DERIVATION OF $\bar{E}$

As mentioned earlier, Planck concentrated on the energy states of resonators in the cavity walls and used the condition that the resonators and cavity radiation were in equilibrium to determine the spectral quality of the radiation. By thermodynamic reasoning (and apparently unaware of Rayleigh's derivation), he arrived at the same expression for  $N(f)$  as Rayleigh. However, Planck arrived at a different form for  $\bar{E}$  by postulating *discrete* values of energy for his resonators. Because it is simpler and more direct, we shall continue with Rayleigh's point of view and *quantize the energy of the standing electromagnetic waves or modes in the blackbody cavity* to derive Planck's law. It is interesting, however, that historically Planck quantized only the material resonators in the cavity and maintained the classical wave nature of light in the cavity.

Instead of Rayleigh's assumption that a standing wave can have any energy from 0 to  $\infty$ , we say that a standing wave of frequency  $f$  can have only *discrete energies*  $E = nhf$  where  $n = 0, 1, 2, 3, \dots$ . We again apply the Boltzmann formula to find the probability that an oscillator is in its  $n$ th state, where now the probability of occupancy is  $P(n) = P_0 e^{-nhf/k_B T}$ . The expression for the average energy of a standing wave, Equation 3.16, now becomes

$$\bar{E} = \frac{\sum_{n=0}^{\infty} (nhf) P_0 e^{-nhf/k_B T}}{\sum_{n=0}^{\infty} P_0 e^{-nhf/k_B T}} \quad (3.47)$$

Using

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

gives

$$\sum_{n=0}^{\infty} e^{-nhf/k_B T} = \frac{1}{(1 - e^{-hf/k_B T})}$$

Substituting this into Equation 3.47 yields

$$\bar{E} = hf(1 - e^{-hf/k_B T}) \sum_{n=0}^{\infty} n e^{-nhf/k_B T} \quad (3.48)$$

To evaluate the sum, we let  $\alpha = hf/k_B T$ , and note that

$$\sum_{n=0}^{\infty} n e^{-n\alpha} = -\frac{d}{d\alpha} \sum_{n=0}^{\infty} e^{-n\alpha} = -\frac{d}{d\alpha} \left( \frac{1}{1 - e^{-\alpha}} \right) = \frac{e^{-\alpha}}{(1 - e^{-\alpha})^2}$$

Therefore,

$$\sum_{n=0}^{\infty} n e^{-nhf/k_B T} = \frac{e^{-\alpha}}{(1 - e^{-\alpha})^2}$$

and the expression for  $\bar{E}$  becomes

$$\bar{E} = \frac{hf e^{-hf/k_B T}}{1 - e^{-hf/k_B T}} = \frac{hf}{e^{hf/k_B T} - 1} \quad (3.49)$$