Size-Dependent Transition to High-Dimensional Chaotic Dynamics in a Two-Dimensional Excitable Medium

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The spatiotemporal dynamics of an excitable medium with multiple spiral defects is shown to vary smoothly with system size from short-lived transients for small systems to extensive chaos for large systems. A comparison of the Lyapunov dimension density with the average spiral defect density suggests an average dimension per spiral defect varying between 3 and 7. We discuss some implications of these results for experimental studies of excitable media. [S0031-9007(98)05536-7]

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Much research in the nonequilibrium physics of excitable media has been motivated by the observation of dynamical states containing defects, i.e., spiral waves in two space dimensions or spiral filaments in three dimensions [1]. Experimental studies in surface oxidation experiments [2] and in fibrillating hearts [3] suggest that many such defects may coexist in dynamically complex states. Although similar states have been reproduced in computer simulations [4–6], there has not yet been careful quantitative analysis of whether the long-time dynamics of such media can be chaotic and, if so, how the properties of this chaos may be related to the statistics of defects, to the size of the medium, and to intrinsic medium parameters. Detailed analysis of mathematical models of excitable media may thus provide new insights in how to analyze spatially extended excitable media, possibly including fibrillating cardiac tissue.

In this Letter, we study numerically a two-dimensional model of a homogeneous excitable medium with an emphasis on determining when spatiotemporal chaos occurs, and on quantitatively analyzing basic time and length scales of observed chaotic states. We use a model introduced by Bär *et al.* [4], as a reduced description of carbon monoxide oxidation on a surface [7], because of its numerical simplicity and because of prior work suggesting the existence of chaos [8]. Extending some recent work by other researchers [4,5,9], we find that the dynamics is strongly dependent on the system size L. For small L, all initial conditions studied rapidly decay to an asymptotic constant or periodic state. As the system size increases, however, the fraction of initial conditions leading to sustained, nonperiodic dynamics increases smoothly, and we discuss the transition from periodic to nonperiodic dynamics with increasing system size. The nonperiodic dynamics sustained in sufficiently large systems are statistically stationary, and we compute Lyapunov exponents and dimensions, defect statistics, and two-point correlation lengths to characterize these states. These different statistics are compared, both to test previous conjectures about the relationship of these correlation lengths [10] and to evaluate the complexity of the defects. Our results indicate that a two-dimensional excitable medium of moderate size with few defects on average can already sustain extensive, high-dimensional chaotic dynamics, a fact with important implications for control of excitable media by small parameter perturbations [11]. In the following, we explain the model, summarize our calculations, and discuss the implications of our results.

The Bär model describes the interaction of an activator field $u(t,x,y)$ with an inhibitor field $v(t,x,y)$ via the partial differential equations

$$\frac{\partial u}{\partial t} = \nabla^2 u + \frac{1}{\epsilon} u(1-u)\left(u - \frac{v+b}{a}\right), \quad \text{(1a)}$$

$$\frac{\partial v}{\partial t} = f(u) - v, \quad \text{(1b)}$$

which we solve numerically in a square domain of side L with either biperiodic (BP) or no-flux (NF) boundary conditions on the field $u(t,x,y)$. The function $f(u)$ has the form

$$f(u) = \begin{cases} 0, & \text{if } u \le 1/3, \\ 1 - 6.75u(u-1)^2, & \text{if } 1/3 \le u \le 1, \\ 1, & \text{if } u > 1, \end{cases} \quad (2)$$

so that the production of the inhibitor v is "delayed" until u exceeds $1/3$. The nonlinear form Eq. (2) leads to three fixed points, one stable and two unstable; the larger unstable fixed point (u^*, v^*) (which does not appear in the widely used Fitzhugh-Nagumo model) seems necessary for the occurrence of spatiotemporal chaos. The parameter ϵ in Eq. (1a) determines the ratio of time scales of the fast field u and slow field v and is the key bifurcation parameter in this paper. The positive parameters a and b were fixed at the values $a = 0.84$ and $b = 0.07$ to take advantage of substantial earlier work using these values [4,8]. Spiral solutions are then known empirically to be unstable when ϵ exceeds a critical value $\epsilon_c \approx 0.069$ [4]. The mechanism of this instability, meander of the spiral core into a branch of the spiral, is apparently unique to

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models with delayed inhibitor production like that given by Eq. (2). In particular, this is not the mechanism of breakup observed in models of cardiac tissue [5,6]. Breakup leads to long-lived, complicated dynamics for certain initial conditions when $\epsilon > \epsilon_c$; a snapshot of such a disordered nonperiodic state with 31 spiral defects is shown in Fig. 1.

Our calculations involved integrating Eqs. (1), calculating the Lyapunov spectrum of the numerical trajectory, and counting the number of spiral defects at successive times. For both kinds of boundary conditions, Eqs. (1) were solved numerically by first introducing second-order accurate finite difference approximations for the spatial derivatives on a uniform square mesh of spacing Δx and then using an algorithm proposed by Barkley [12]. For the calculations reported below, we used a spatial grid size $\Delta x = 0.50$ and time step $\Delta t = 0.05\epsilon$. The spectrum of Lyapunov exponents λ_i and the Lyapunov fractal dimension D were calculated by well-known algorithms based on linear variational equations [13] that were integrated by a forward-Euler algorithm with the same grid and time step. The time step chosen, $\Delta t = 0.05\epsilon$, was much smaller than that required by integration of only Eqs. (1), but was found to be necessary to compute Lyapunov exponents accurately to within a few percent [14]. For given boundary conditions and initial data, Eqs. (1) were integrated for 2000 time units to allow a statistically stationary state to be obtained, and then the full system with variational equations was integrated for an additional 1000 time units ($\approx$200 spiral periods), during which statistics were calculated.

To study the dependence of the dynamics on initial conditions, we integrated Eqs. (1) from each of 100 initial conditions generated by distributing the field values uniformly (at each grid point) in the ranges $u \in [0.8u^*, 1.2u^*]$, $v \in [0.8v^*, 1.2v^*]$. This procedure was repeated for both boundary conditions and for square systems of side length L varying from 5 to 40. For all initial conditions, the dynamics was short-lived in small systems [$L < 15$ (for NF boundary condition) or $L < 8$ (for BP)], decaying in less than 100 time units to either the stable uniform state or to a plane-wave state (only in the case of biperiodic boundary conditions). Sufficiently large systems ($L > 35$) sustained dynamics for at least 3000 time units. The fraction f of initial conditions which led to nonperiodic dynamics sustained for a time T (either 100 or 1000 time units) is shown in Fig. 2 as a function of system size for both boundary conditions. For biperiodic boundary conditions [Fig. 2(a)], the curve is independent of the cutoff time T, indicating that transients either die quickly or are sustained indefinitely (more than 50 000 time units). For no-flux boundary conditions, all initial conditions studied eventually decayed to a stationary state; the mean and median transient times both scale exponentially with system size [14], much like the supertransient behavior observed previously in some one-dimensional systems [15]. These results suggest that excitable media of intermediate size may have an appreciable basin of attraction both for nonperiodic dynamics and for periodic or constant dynamics. Whether

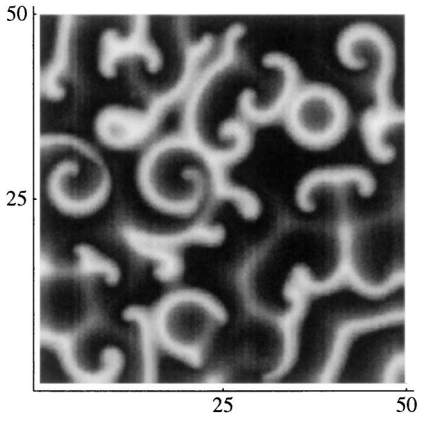

FIG. 1. Density plot at time $t = 500$ of the slow field $v(t, x, y)$ for a spatiotemporal chaotic state with 31 spiral defects present. Dark and light regions correspond, respectively, to values less and greater than the value $v^* = 0.484$ corresponding to the unstable fixed point; the field values span the range $v \in [0, a - b]$. Parameter values were $\epsilon = 0.074$, $a = 0.84$, $b = 0.07$, $L = 50$, $\Delta x = 0.5$, and $\Delta t = 0.0037$.

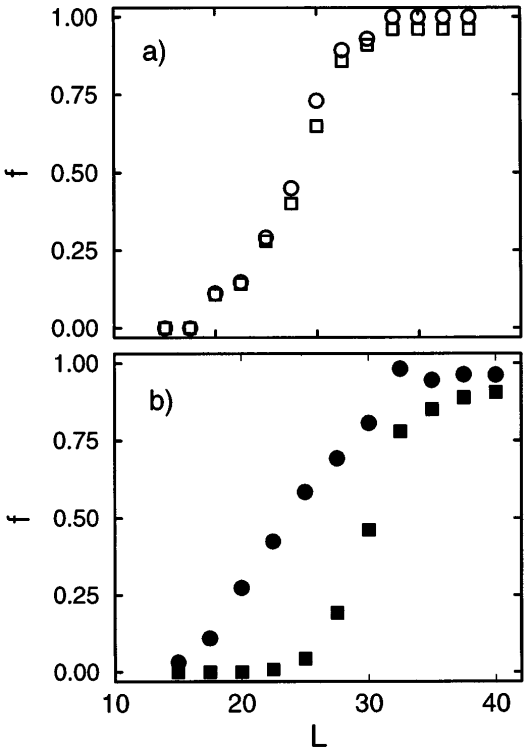

FIG. 2. Fraction f of 100 random initial conditions still exhibiting nonperiodic dynamics after a given time. (a) For biperiodic boundary conditions with cutoff time $T_{np} = 100$ (circles), 1000 (squares), or any larger value, the transition has the same form, with systems of side length $L > 25$ nearly always exhibiting sustained dynamics. (b) For no-flux boundary conditions with cutoff time $T_{np} = 100$ (circles) and $T_{np} = 1000$ (squares), the median transient time depends on the cutoff. Comparison of these graphs shows that dynamics are substantially less likely to be sustained for a given time with no-flux boundary conditions. The parameters used were the same as in Fig. 1.

this accounts for the observation that fibrillation *sometimes* occurs in hearts of intermediate size remains unclear, both because of the differing breakup mechanisms and because of the effect of the third dimension in heart tissue [9,16].

The fact that a given state was transient was revealed only by an eventual abrupt change to the uniform state; the dynamics of the transient itself was found to be statistically stationary. For parameter values $\epsilon > \epsilon_c$ and for system sizes $L > 25$, these statistically stationary states were found to be high dimensional ($D \geq 20$) and extensively chaotic [17] as shown in Fig. 3 by a linear dependence of D on L^2. From the asymptotic slopes of the curve in Fig. 3, an intensive dimension density $\delta = \lim_{L^2\to\infty} \partial D/\partial(L^2)$ was obtained and then reexpressed as a dimension correlation length $\xi_\delta = \delta^{-1/d}$ for a $d = 2$ dimensional domain [17]. To test a speculation of Bayly *et al.* [10] that knowledge of the experimentally accessible two-point correlation length ξ_2 might provide knowledge of the dynamical length ξ_δ for a chaotic state of spiral defects, we computed ξ_2 and ξ_δ for several values of the parameter ϵ. For each ϵ value studied, the two-point correlation function $C(r)$ had a similar monotonically decreasing but nonexponential form so we estimated ξ_2 by the position of the first zero crossing of $C(r)$ [14]. As shown in Fig. 4(a), the two lengths agree within a factor of 1.5 or better but have opposing trends as ϵ increases (from 0.07 to 0.12), with ξ_δ decreasing and ξ_2 increasing slightly. It is unclear from these data whether an estimate of ξ_δ can, in this medium, be obtained by measuring ξ_2. In any case, further analysis of more physiologically accurate models will be needed to relate ξ_δ and ξ_2 for the heart data of Bayly *et al.* [10].

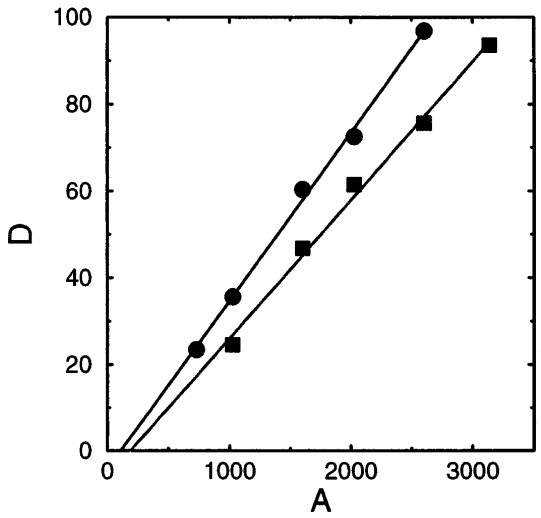

FIG. 3. Lyapunov dimension D versus system area $A = L^2$ of Eqs. (1) for the parameter values of Fig. 1. Extensive (linear) scaling is found for two different boundary conditions, no-flux (squares) and periodic (circles). Data for $L \leq 25$ did not exist since all initial conditions decayed quickly to the uniform state. The dimension extrapolates to zero for a positive system size, so the ratio D/L^2 of the dimension to system area asymptotes slowly to the dimension density δ.

We also explored whether the fractal dimension D of the chaotic states was related to the statistics of the number $N(t)$ of spiral defects, e.g., to its time average $\langle N\rangle$. The spirals were counted at successive times by locating their cores, which occur at those points (x, y) in the medium where the fields (u, v) take on the values (u^*, v^*) [4]. It has been shown previously that $N(t)$ is constant for $\epsilon < \epsilon_c$, in which case the time average $\langle N\rangle$ is fixed by the choice of initial condition [4]. We found that the mean $\langle N\rangle$ scaled extensively with system area L^2 for both boundary conditions considered, so that the average defect density $n = \langle N\rangle/L^2$ was independent of system size [8]. The ratio $d = D/\langle N\rangle$ of the Lyapunov dimension to the mean number of defects therefore defines an intensive quantity that measures the number of dynamical degrees of freedom associated with each defect on average. Because the extensive quantities D and $\langle N\rangle$ are of the form $\alpha L^2 + \beta$ rather than simply αL^2 (where α and β are constants), the ratio d asymptotes slowly to a constant value close to δ/n. We studied the dependence of $D/\langle N\rangle$ on area L^2 for two different values of ϵ, and found that we could estimate the ratio d accurately using a single system of side length $L = 40$ with periodic boundary conditions. For

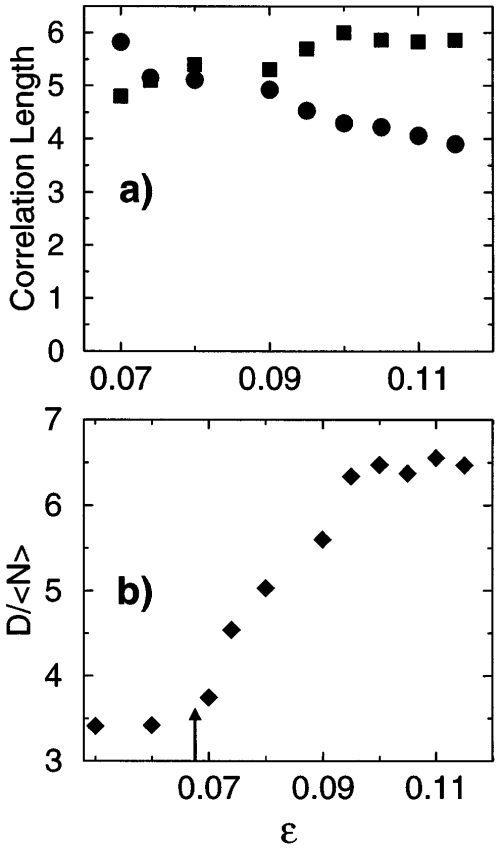

FIG. 4. (a) Dimension correlation length ξ_δ (circles) and two-point correlation length ξ_2 (squares) for different ϵ values. (b) Degrees of freedom per mean defect, $D/\langle N\rangle$, as a function of ϵ. This ratio increases steadily with ϵ above the transition to chaos at $\epsilon = \epsilon_c$ (marked by the arrow), and varies little in the region $\epsilon > 0.095$.

this system size, d increases smoothly with ϵ from less than 4 for $\epsilon \approx \epsilon_c$ to nearly 7 for $\epsilon \geq 0.095$ at which point it becomes approximately constant [see Fig. 4(b)]. Thus a fixed number of degrees of freedom cannot, in this manner, be associated with each spiral defect in an excitable medium.

In summary, for a particular model of a two-dimensional excitable medium [4,8], we have demonstrated by numerical calculations a smooth transition from short-lived transient dynamics to extensive, high-dimensional chaotic dynamics with increasing system size L for both no-flux and biperiodic boundary conditions. Small systems never exhibited sustained chaotic dynamics, but nonperiodic dynamics in sufficiently large systems was found to be statistically stationary on such long time scales and for such a large fraction of random initial conditions that Lyapunov spectra of the dynamics converged well to a value independent of the initial condition. Although previous results based on time series allowed computation of one Lyapunov exponent [18,19], we computed enough Lyapunov exponents to determine the Lyapunov dimension D in an excitable medium with many spiral defects. We found no clear relation between the dimension correlation length ξ_δ and the widely used two-point length ξ_2; however, the numerical similarity of these lengths supports a previous conjecture that $\xi_\delta \approx \xi_2$ in data taken on fibrillating hearts [10]. The mean Lyapunov dimension per defect of 3 to 7 suggests that defects are more complicated than in the defect-turbulent regime of the complex Ginzburg-Landau equation [20], and that the dynamics of excitable media with even a few defects may be quite high dimensional. This high dimensionality in turn suggests that it will be difficult to stabilize such states by small variations of parameters [11], and may explain why some previous attempts to analyze the dynamics of fibrillation with low-dimensional time series embedding techniques have been inconclusive [21]. The unusual nature of the spiral-wave breakup in this medium leaves to future work the important question of whether the results obtained here apply to other excitable media, including fibrillating ventricles.

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